Helicoids, Wrinkles, and Loops in Twisted Ribbons

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We investigate the instabilities of a flat elastic ribbon subject to twist under tension, and develop an integrated phase diagram of the observed shapes and transitions. We find that the primary buckling mode switches from being localized longitudinal along the length of the ribbon to transverse above a triple point characterized by a crossover tension which scales with ribbon elasticity and aspect ratio. Far from threshold, the longitudinally buckled ribbon evolves continuously into a self-creased helicoid with focusing of curvature along triangular edges. Further twist causes an anomalous transition to loops compared with rods due to the self-rigidity induced by the creases. When the ribbon is twisted under high tension, transverse wrinkles are observed due to the development of compressive stresses with higher harmonics for greater width to length ratios. Our results can be used to develop functional structures using a wide range of elastic materials and length scales.

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Finding smart, reliable, and efficient strategies toward developing functional shapes at smaller and smaller scales has been a focus of intense research in elastic materials ranging from synthetic yarns to atomically thin graphene sheets and soft tissue. Although buckling, wrinkling, and creasing of thin sheets under compressive loading has been widely studied recently, the instabilities and shapes observed under the application of a twist have received far less attention. Seminal analysis by Green, confirmed later by numerical simulations, showed that a ribbon under tension is unstable upon twist due to the development of a spatially nonuniform compressive stress. Under finite tensile load, transverse buckling and localized loop formation has been shown using numerical simulations. In the case where the ribbon is inextensible, stress localization can occur. However, no previous studies, either theoretical or experimental, provide a comprehensive view of these observations.

Here, we show that the various ribbon shapes are organized in a phase diagram using only two control parameters: the initial tension and the twist angle. In this phase diagram, the lines separating regimes—where helicoids, buckled helicoids, and loops are observed—meet at a triple point which sets the value of a characteristic tension \( T^* \). The primary buckling mode switches from longitudinal to transverse when crossing \( T^* \). Below \( T^* \), we characterize a secondary instability leading to a loop, and show that the development of creases affects this transition dramatically. We find that the transitions studied here are different in nature and more diverse than the ones observed with rods. We provide the first clear experimental observation of the various buckling modes and further develop an integrated understanding of their onset using simple scaling arguments. Our results for the onset of longitudinal buckling compare well to those reported using more elaborate asymptotic analysis, and our approach allows us to further characterize the onset of the transverse buckling and the characteristic tension \( T^* \).

We study a ribbon of length \( L \), width \( W \), and thickness \( h \) which is held under tension \( F \) using clamped boundary conditions at its ends, and then twisted through a prescribed angle \( \alpha \) about its central long axis. Other types of boundary conditions may be considered but they are more difficult to implement experimentally.Ribbons composed of biaxially-oriented polyethylene terephthalate with Young’s modulus \( E = 3.4 \text{ GPa} \) and \( L > 10W > 500h \) are used, unless otherwise stated. The normalized twist angle \( \theta = \frac{\alpha W}{L} \) and the non-dimensional tension \( T = \frac{F}{EhW} \) are used as control parameters to describe the applied conditions as illustrated in Fig. (a). Examples of a helicoid, buckled helicoids in the longitudinal and transverse direction, a creased helicoid, and a localized loop obtained by simply varying \( T \) and \( \theta \) are shown in Fig. (b)-(f), respectively.

Phase diagram - \( T^* \) We obtained the phase diagram by measuring the critical angle for the longitudinal buckling, the loop transition, and the transverse buckling for a ribbon by increasing \( \theta \) in small increments from zero for an initial tension \( T \) (see Fig. (g)). Note that the local deformations are within a good approximation in the linear elastic regime of the material, while plastic deformations are observed only near the edge of the ribbon for high twist angle (\( \theta > 0.5 \)). Interestingly, we find that the three lines, corresponding to the onset of the three instabilities mentioned above, meet at a triple point characterized by the crossover tension \( T^* \). Longitudinal buckling occurs below \( T^* \) and the critical angle \( \theta_L \), increases with \( T \). Above \( \theta_L \), the postbuckling shape evolves progressively into a creased helicoid with folds of alternating angles. Upon further twist, a secondary instability occurs at a critical angle \( \theta_F \) where the ribbon shape transforms into a loop with self-contact. The nature of this transition depends on \( T \). Below \( T^* \), the ribbon switches dynamically into a loop configura-
tion with $\theta_T$ decreasing linearly with $T$. Above $T^*$, the buckling is now transverse with a critical angle, also denoted by $\theta_T$, independent of $T$ over the range of tension explored. Further twisting can lead to loops with self-contact. More complicated self-wrapped configurations can occur for other ribbon aspect ratios and are not discussed here in the interest of space. Also, it may be noted that changing the boundary conditions affects the phase diagram but investigating this is beyond the scope of the present study as we are primarily interested in clamped edges.

**Buckling and postbuckling - Below $T^*$** In order to obtain quantitative information on the shape of the ribbon, we use a Varian Medical Systems micro-focus X-ray Computer Tomography instrument which allows ribbons with lengths up to 15 cm to be fully scanned to resolutions within tens of microns. The mean curvature $H$ and the Gaussian curvature $K$ are then obtained which characterize the shapes. Figure 2(a)-(c) shows the map of the curvature superimposed on the ribbon before and after buckling. A helicoid shape can be expected if a constant twisting rate along the ribbon is assumed which is confirmed by a zero mean curvature (Fig. 2a)). When $\theta = \theta_L$, the ribbon buckles longitudinally (Fig. 2b)) with a pattern essentially localized in the central portion of the ribbon, which on further twist can develop creases where the curvature is further localized along the sides of a triangle (Fig. 2c)).

To understand this mode of buckling, we consider the non-dimensional longitudinal stress of a thin filament at a distance $y$ from the central axis given by $\sigma_L = \frac{1}{2} \theta^2 (y/W)^2$, where $\sigma_L$ is the stress of the central line. Then, $\sigma_L$ is set by the condition of constant load $(\sigma) = T = \sigma_L + \theta^2/24$, where $\langle .. \rangle$ stands for the average along the transverse direction. Introducing $\theta_0 = \sqrt{24T}$,

$$\sigma_L = \frac{1}{24} \left( \theta_0^2 - \theta^2 \right).$$

(1)

Plotting $\theta_L$ in Fig. 2d), we note that buckling occurs consistently above $\theta_0$ when compressive stress develops longitudinally. Just above $\theta_0$, the width of the compressive region of the ribbon is $W_c \approx W \sqrt{\frac{2 \theta - \theta_0}{3 \theta_0}}$ and $\sigma_L \approx \frac{\theta_0}{12} (\theta - \theta_0)$.

Now, to develop the criteria for longitudinal buckling as a result of the compressive stresses, we use energy arguments by considering only the central part of the ribbon where quadratic dependence of the longitudinal stress with $y/W$ can be neglected. Let us consider a buckled mode characterized by a typical amplitude $A$ and wavelength $\lambda$. Then, the change in the elastic energy
The longitudinal buckling wavelength \( \lambda_L \) with finite thicknesses, \( \sigma \), \( \partial \sigma \), unstable buckling mode characterized by the condition \( \sigma \theta \) and \( \partial \sigma \) above \( \theta \), we find that \( \sigma \theta \) increases linearly with \( h/W \) (Fig. 2(e)) consistently with Eq. 4. The corresponding measured \( \lambda_L \) is observed to decrease as a function of \( T \) as shown in Fig. 2(f) consistent with Eq. 5 with a numerical prefactor of 3.8. Further, the experimentally obtained prefactors for \( \theta \), and \( \lambda_L \) are consistent with those seen numerically [12].

Keeping the tension below \( T < T^* \), and twisting the ribbon far above the threshold for longitudinal buckling, the pattern breaks the right-left symmetry by progressively localizing the mean curvature along lines of alternating angle until self-creased helicoidal structure forms as shown in Fig. 2(c). The sharpness of the creases is more pronounced at lower tension. This created helicoidal pattern, which is well known in origami, was demonstrated previously for inextensible ribbons [16], but no satisfactory explanation for the pattern selection is available. We find the size of the triangle is not unique and is given by the wavelength at onset as shown in Fig. 2(b) and (c) suggesting that our stability analysis is actually a selection mechanism for the observed pattern. Further, we find that the creasing helicoid can be observed in a regime of higher tension (\( \theta \theta \) \( h/W \)) than previously thought. The condition of inextensibility, usually required to form localized stressed regions in thin elastic plate, may be locally met within the central compressed region of the ribbon.

Transverse buckling - Above \( T^* \) Transverse buckling is observed above \( T^* \) with buckling patterns that depends mainly on geometrical parameters. When \( L \gg W \), only the fundamental mode is observed (Fig. 3(a)) which has been seen numerically [14]. Further twisting leads to a localized loop with a unique self-contact (Fig. 3(g)). However, higher modes can be obtained by decreasing \( h/W \) and \( L/W \) as shown in Fig. 3(b)-(c) where transverse cross-sections of ribbons (\( x = 0 \)) twisted above \( \theta T \) can be seen. Further twisting leads to shapes with a complex set of self-contacts. We measure the critical angle for various combinations of geometrical parameters \( L/W \) and \( h/W \) for \( T > T^* \) and only consider critical twist angle \( \theta T > 2\pi/L/W \) where the fundamental buckling mode is observed. Figure 3(d) shows \( \theta T \) versus \( L/W \), and where each symbol corresponds to a different length. In Fig. 3(e), we plot \( \theta T \) as a function of
h/W and find that all the data collapse onto a single line \( \theta_T = 4.4(h/W)^{1/2} \). This trend can be explained by a stability analysis of a thin transverse section of the ribbon under compression. A compressive transverse stress scaling as \( \sigma_T \sim E\theta^h \) has been argued using scaling arguments [13]. The section buckles when \( \sigma_T \sim E(h/W)^2 \), which corresponds to the buckling condition for a beam of length \( W \) and thickness \( h \). This condition leads to a critical angle \( \theta_T \sim \sqrt{h/W} \) that only depends on a geometrical parameter unlike \( \theta_L \), and is consistent with our experimental data (see Fig. 3(e)). Plastic deformations can be observed near the edges for \( \theta > 0.5 \) and will have to be taken into account in subsequent analysis for a more precise prediction, although the scaling is expected not to change.

**Secondary instability - Below \( T^* \)** We next discuss the anomalous secondary transition to the loop in more detail (see Fig. 4). While increasing \( \theta \) above \( \theta_T \) (solid blue line), the ribbon dynamically jumps to a loop configuration with self-contact. When \( \theta \) is then decreased below \( \theta_T \), the loop remains stable demonstrating the hysteretic nature of the transverse buckling below \( T^* \). Upon further decrease, the ribbon is no longer in self-contact below the dashed red line, and the helicoid is recovered below the solid red line. This transformation from a shape with a negative Gaussian curvature (helicoid) to a cylindrical shape (loop) is a way for a ribbon to release part of its in-plane stress. But, surprisingly, the critical angle decreases linearly with the tension. This feature is in sharp contrast with the higher \( T \) regime. It means that a loop is formed more easily at higher tension unlike rods where the tension is reduced to trigger the loop transition [19]. The self-rigidity that arises from the formation of creases with large curvature have a great resistance against deformation. However, in the case of a ribbon, the rigidity mechanism is spontaneous because no intrinsic curvature was present in the pre-stressed flat ribbon.

**Scaling for \( T^* \)** Now, by using the condition \( \theta_L = \theta_T \), we can obtain a scaling for the crossover tension \( T^* \). Substituting the expressions for the critical angle for both buckling modes in the thin ribbon limit \( (h/W \rightarrow 0) \), we obtain \( T^* \approx 0.8h/W \). For the ribbon used in Fig. 4(e), our scaling analysis gives \( T^* \approx 4 \times 10^{-3} \), in good agreement with our experimental data.

Fundamental questions on the onset of buckling, wrinkling and crumpling of thin elastic materials and post-buckling behavior is still a matter of significant debate [21]. Because rarely has a single experiment embraced such a diversity of shapes and behaviors without
the need for a complex coupling with a substrate \[22, 23\], or frictional walls \[24, 25\], the twisted ribbon configuration offers a new paradigm to understand the strong nonlinear and singular elastic theory as captured by the Föppl-von Kármán equations.

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\[18\] See Supplemental Material at [] for mechanical characterization of the material.
\[20\] B. Audoly and Y. Pomeau, Elasticity and Geometry (Oxford University Press, 2010).