Math 105 History of Mathematics

First Test Answers

Prof. D. Joyce, October, 2010

Scale. 89–101 A. 78–88 B. 58–77 C. Median 84.

Problem 1. Essay. [25] Select one of the three topics A, B, and C.

Topic A. Explain the logical structure of the *Elements* (axioms, propositions, proofs). How does this differ from earlier mathematics of Egypt and Babylonia? How can such a logical structure affect the mathematical advances of a civilization?

The *Elements* begins with definitions and axioms. Some of these just describe the terms to be used, others are more substantive and state specific assumptions about properties of the mathematical objects under study. Definitions are given for new concepts stated in terms of the old concepts as the new concepts are needed. Propositions are stated one at a time only using those terms already introduced, and each proposition is proved rigorously. The proof begins with a detailed statement of what is given and what is to be proved. Each statement in the proof can be justified by previous statements, axioms, previously proved propositions, or as an assumption in the beginning of a proof by contradiction. The last statement in a proof is that which was to be proven.

The preHellenic mathematics of Egypt and Babylonia consisted of tables and of solutions to problems of various types. The solutions were meant to describe the methods to solve the problems, what we now call algorithms. No indication was given why the methods should work.

A strict logical structure is primarily needed to convince the audience of the validity of the theory, but it has other purposes. It is used to find flaws in arguments, and even in previously accepted statements. It can be used to find hidden assumptions. More importantly, adhering to a strict logical structure suggests new concepts and new results. Egyptian mathematics reached its high point early in the history of Egypt, about 2000 BCE, and did not progress past that. Babylonian mathematics reached its high point about the same time, 1800 BCE, the Old Babylonian Empire, and also failed to progress after that. The mathematics of both cultures was directed to solving problems. Greek mathematics, on the other hand, progressed as logic developed to the time of Euclid, and continued to progress for several centuries as we will see.

Topic B. Compare and contrast the arithmetic of the Babylonians with that of the Egyptians. Be sure to mention their numerals, algorithms for the arithmetic operations, and fractions. Illustrate with examples. Don't go into their algebra or geometry for this essay.

Here's an list of topics that could go into an essay, but they're not all needed in an essay: Numerals: Egypt — base 10, hieroglyphics repeated symbols for 1, 10, 100, etc, hieratic abbreviations; Babylonia – base 60, positional numeration but used only two marks. Addition and subtraction similar to ours in both cultures. Multiplication: Egypt — repeated duplication two columns (give example); Babylonia — long multiplication similar to ours. Division: Egypt — virtually the same as multiplication, but order of selecting rows different; Babylonia — use a table of reciprocals then multiply. Square roots: Egypt — unknown; Babylonia — tables and fast algorithm. Fractions: Egypt — unit fractions, doubling table for fractions, extend multiplication and division algorithms to fractions; Babylonia — positional numeration like our decimal fractions, but rarely indicate where the decimal point is, same arithmetic algorithms as for whole numbers, but complicated by mentally keeping track of decimal point. One possible essay summary: Babylonians had much more efficient notation and algorithms, could deal with fractions much better.

Topic C. Aristotle presented four of Zeno's paradoxes: the Dichotomy, the Achilles, the Arrow, and the Stadium. Select one, but only one, of them and write about it. State the paradox as clearly and completely as you can. Explain why it was considered important. Refute the paradox, either as Aristotle did, or as you would from a modern point of view.

One of the arguments, the Stadium, depends on conceiving a line as being made out of points in a row, one next to another. Even in Euclidean geometry, there are points on a line, but they do not have that arrangement. The Arrow also depends on time being composed of instants, but not explicitly arranged in a row, one next to another. It does depend, however, in assuming that motion can be determined at an instant without looking at positions at other instants. Aristotle refutes these paradoxes by denying lines are composed of points and time of instants, and by allowing motion only over an interval of time. Aristotle's solution to these two paradoxes differs from that of modern mathematics.

The Dichotomy and the Achilles assert an infinite sequence of occurrences in a finite amount of time. The arguments leading to these occurrences are different in the two paradoxes, but Zeno apparently denied an infinite number of instants in a finite interval of time. Aristotle and the modern point of view agree here. There is no paradox in assuming that there are an infinite number of points on a finite line, or that there are an infinite number of points in time in a finite interval of time.

See the text for more details.

Problem 2. [15] Find the greatest common divisor of the two numbers 11484 and 7902 by using the Euclidean algorithm. (Computations are sufficient, but show your work. An explanation is not necessary.)

For the Euclidean algorithm repeatedly subtract the smaller number from the larger to get smaller and smaller numbers until the smaller divides the larger. Subtract 7902 from 11484 to get 3582. Then work with 7902 and 3583. Now you can subtract 3582 from 7902 a couple times to get a remainder of 738. (That's the same as dividing 7902 by 3582 and keeping the remainder.) Then subtract 738 repeatedly from 3582 to get a remainder of 630. Then $738 - 630 = 108$, and 108 repeatedly subtracted from 630 gives 90, and $108 - 90 = 18$. Finally, 18 divides 90, so 18 is the greatest common divisor.

Problem 3. [24] On Eudoxus' definition of equality of ratios of magnitudes. Answer each part in a couple of sentences.

a. [12] The Pythagorean philosophy of numbers was summarized by the Pythagorean Philolaus as

> All things which can be known have number; for it is not possible that without number anything can be either conceived or known.

Explain why the discovery of incommensurable magnitudes (such as the side of a square and its diagonal) led to a crisis for the Pythagoreans.

To Philolaus and the Pythagoreans, "number" meant whole positive whole number. If the philosophy was correct, shouldn't there be a unit which measures both the diagonal and side of a square evenly, that is, some positive whole number of units make up the diagonal while another positive whole number of units make up the side? But that can't be done.

b. [4] Explain in your own words Eudoxus definition of equality of ratios, that is, when is the ratio $a : b$ of two magnitudes of one type equal to the ratio $c : d$ of two magnitudes of (possibly) another type.

Whenever some multiple na of a is greater than some other multiple mb of b, then the first multiple nc of c will be greater than the second multiple md of d. But if $na = mb$, then nc will equal md. And if $na < mc$, then nc will be less than md. And these statements must hold for *all* whole positive numbers m and n.

(Note that Eudoxus did not say that $a : b = c : d$ when $ad = bc$. That's a useful definition in some cases, for example when a, b, c, and d are all lines; then $ad = bc$ says one rectangle equals another rectangle. It also works when a and b are lines while c and d are plane figures; then $ad = bc$ says one solid equals another solid. But it doesn't work when they're all plane figures because ad would refer to some 4-dimensional object, a concept beyond the ancient geometers' comprehension.)

c. [4] Using this definition, show that the numeric ratio 3:5 does not equal the ratio 4:6.

All we have to do is find values for m and n so that $3n$ compares to $5m$ in a different way than $4n$ compares to $6m$. There are lots of choices that work. For instance, $(m, n) = (3, 5)$, for then $3 \cdot 5 = 5 \cdot 3$ but $4 \cdot 5 > 6 \cdot 3$. For another example, take $(m, n) = (7, 11)$, then $3 \cdot 11 < 5 \cdot 7$ but $4 \cdot 11 > 6 \cdot 7$.

d. [4] Explain how this definition resolved the crisis and supported the Pythagorean philosophy.

With Eudoxus definition, ratios like the diagonal to the side of a square can be known even if they're not commensurable, and they can be known by means of whole numbers.

Problem 4. [16] On areas of circles. The cultures we have studied—Egyptian, Babylonian, and Greek—all knew how to approximate the area of a circle. Choose one of the cultures and describe one method that was used to compute the area of a circle. Your description should only be a sentence or two long. Illustrate the method by detemining the area of a circle whose diameter is 9 cubits. (A cubit being a measure of length, the length of a forearm, used by all three cultures.)

One Egyptian method that was quite accurate was to take the square on $\frac{8}{9}$ of the diameter. In this case, that's 64 square cubits.

The Babylonians often used a method that we can summarize as approximating π by 3, equivalently, the area of a circle is approximately $\frac{3}{4}$ the area of the square on the diameter. In this case, that's $\frac{3}{4}$ of 81, or 60 plus $\frac{3}{4}$, or, as the Egyptians would have expressed it, 60 plus $\frac{1}{2}$ plus $\frac{1}{4}$.

Various values were used by the Greeks to approximate π, Archimedes' $\frac{22}{7}$ being one of them. With that approximation, the area of this circle is $\frac{22}{7}(\frac{9}{2})^2 = \frac{891}{14} 63\frac{9}{14}$.

Problem 5. [25; 5 points each part] True/false.

a. Euclid's parallel postulate (Postulate 5 in Book I of the *Elements*) stated that lines in the same direction are parallel. False. That's nothing like the parallel postulate.

b. The ancient Babylonians knew the Pythagorean theorem (the square on the hypenuse of a right triangle is equal to the sum of the squares on the other two sides) over a thousand years before Pythagoras. True.

c. Each of the propositions in Euclid's *Elements* includes a proof. True.

d. A triangular number is the perimeter of an equilateral triangle, for example, 15 is a triangular number since an equilateral triangle of side length 5 has perimeter 15. False.

e. Whereas Egyptians used common fractions like $\frac{2}{5}$, Babylonians prefered unit fractions like one-third plus one-fifteenth. False.