Assignment 5 answers
Math 105 History of Mathematics
Prof. D. Joyce, Spring 2013

The assignment. Chapter 3, page 91, exercises 27 and 36; and Chapter 4, page 127, exercises 1, 2, and 3.

Chapter 3. Exercise 27. Construct geometrically the solution of $8 : 4 = 6 : x$.

For this, assume that the 8, 4, and 6 are lengths of lines, and we're to construct a line of length x to satisfy the condition.

There are lots of ways to do this, and all of them depend on similar triangles in some way or other. For instance, you could use the diagram in problem 7, that is, Euclid's proposition I.44. Or you could use Euclid's proposition VI.12 which has only the essential elements of the construction. There are other figures that would do as well.

Exercise 36. Use Euclid's criterion in Proposition IX.36to find the next perfect number after 8128.

The proposition says, in modern notation, that if $2^n - 1$ is prime, then $(2^n - 1)2^{n-1}$ is perfect. So the problem is to find the next prime 1 less than a power of 2. The first four such primes are $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 31$, and $2^7 - 1 = 127$. Therefore, the four perfect numbers they generate are $3 \cdot 2 = 6$, $7 \cdot 4 = 28$, $31 \cdot 16 = 496$, and $127 \cdot 64 = 8128$. Our job is to find the next one.

The next few numbers of the form $2^n - 1$ are not prime. $2^8 - 1 = 255$ is divisible by 5. $2^9 - 1 = 511$ is divisible by 7. $2^{10} - 1 = 1023$ is divisible by 11. $2^{11} - 1 = 2047$ is divisible by 23. $2^{12} - 1 = 4095$ is divisible by 5. Finally, $2^{13} - 1 = 8191$ is prime. Therefore, $8191 \cdot 4096 = 33550336$ is the next perfect number.

Chapter 4. Exercise 1. Find where to place the fulcrum in a lever of length 10 m so that a weight of 14 kg at one end will balance a weight of 10 kg at the other.

The distance is inversely proportional to the weight, so a weight of 10 kg will balance the other weight at a distance of 14 kg.

Exercise 2. If a weight of 8 kg is placed 10 m from the fulcrum of a lever and a weight of 12 kg is placed 8 m from the fulcrum in the opposite direction, toward which weight will the lever incline?

If the 12 kg weight were replaced by a 10 kg weight, then like in the previous exercise, it would balance. But 12 kg is greater than 10 kg, so it will incline toward toe 12 kg weight.

Exercise 3. An alternative method by which ARchimedes could have solved the crown problem is given by Vitruvius in *On Architecture*. Assume as in the text that the crown is of weight W, composed of weights w_1 and w_2 of gold and silver, respectively. Assume that the crown displaces a certain quantity of fluid, V. Furthermore, suppose that a weight W of gold displaces a volume V_1 of fluid, while a weight W of silver displaces a volume V_2 of fluid. Show that

$$V = \frac{w_1}{W}V_1 + \frac{w_2}{W}V_2$$

and therefore

$$\frac{w_1}{w_2} = \frac{V_2 - V_1}{V - V_1}.$$

The amount of water displaces by gold is proportional to the weight of gold. Since a weight W displaces V_1 fluid, therefore the weight w_1 of gold in the crown displaces $\frac{w_1}{W}V_1$ fluid. Likewise, a weight w_2 of silver in the crown will displace $\frac{w_1}{W}V_2$ fluid. Thus, the whole crown will displace

$$V = \frac{w_1}{W}V_1 + \frac{w_2}{W}V_2$$

fluid.

There are various algebraic steps to convert that equation to the result. Here's one sequence of steps. Multiply both sides of the equation by W and note that $W = w_1 + w_2$. Then

$$(w_1 + w_2)V = w_1V_1 + w_2V_2.$$

Subtract w_1V_1 and w_2V from both sides of the equation. Then

$$w_1(V - V_1) = w_2(V_2 - V)$$

whence

$$\frac{w_1}{w_2} = \frac{V_2 - V}{V - V_1}.$$

(Note that there was a typo in the statement of the exercise. There was a V_1 where there should have been a V.)

Math 105 Home Page at
`http://math.clarku.edu/~djoyce/ma105/`