First Test Answers
Math 120 Calculus I
September, 2013

Scale. 90–100 A, 80–89 B, 65–79 C. Median 80.

1. [12] On limits of average rates of change. Let \( f(x) = x^2 - 3x \).

a. [4] Write down an expression that gives the average rate of change of this function over the interval between \( x \) and \( x + h \), and simplify the expression.

\[
\frac{f(x + h) - f(x)}{h} = \frac{((x + h)^2 - 3(x + h)) - (x^2 - 3x)}{h}
\]

b. [8] Compute the limit as \( h \to 0 \) of that average rate of change.

\[
\lim_{h \to 0} \frac{((x + h)^2 - 3(x + h)) - (x^2 - 3x)}{h} = \lim_{h \to 0} \frac{2xh + h^2 - 3h}{h} = \lim_{h \to 0} (2x + h - 3) = 2x - 3
\]

2. [10; 5 points each] On the intuitive concept of limit and continuity.

a. [5] Sketch the graph \( y = f(x) \) of a function for which \( \lim_{x \to 0} f(x) \) does not exist.

There are many such graphs. For example, if there’s a jump in the value of \( f \) at \( x = 0 \), then that limit won’t exist. See section 2.4 of the text.

b. [5] Sketch the graph \( y = f(x) \) of a function defined everywhere, the limit \( \lim_{x \to 0} f(x) \) does exist, but \( f \) is not continuous at \( x = 0 \).

This can be achieved by making \( f(0) \) unequal to the limit, but make sure that the function is defined at \( x = 0 \). See section 2.5 of the text.

3. [10; 5 points each property] On asymptotes.

a. Sketch the graph of a function \( f \) such that \( \lim_{x \to 2^+} f(x) = \infty \) and \( \lim_{x \to 2^-} f(x) = -\infty \).

The graph of the function should be asymptotic to the vertical line \( x = 2 \). See section 2.6 of the text.

b. Sketch the graph of a function \( f \) such that \( \lim_{x \to \infty} f(x) = 1 \).

The graph of the function should be asymptotic to the horizontal line \( y = 1 \). See section 2.6 of the text.

4. [28; 7 points each part] Evaluate the following limits. If a limit diverges to \( \pm \infty \) it is enough to say that it doesn’t exist.

a. \( \lim_{x \to 1} \frac{x^2 - 1}{x^2 - 3x + 2} \)

The expression needs to be simplified before taking the limit.

\[
\lim_{x \to 1} \frac{(x - 1)(x + 1)}{(x - 1)(x - 2)} = \lim_{x \to 1} \frac{x + 1}{x - 2} = -2
\]

b. \( \lim_{x \to 1} \frac{x^2 - 4}{x^2 - 3x + 2} \)

The number approaches \(-3\) while the denominator approaches \(0\), so the limit of the quotient doesn’t exist.

c. \( \lim_{x \to \infty} \frac{4x^3 - 2x}{9x^3 + 1} \)

The numerator and denominator have the same degree, so as \( x \to \infty \), the value approaches the ratio of the leading coefficients, \( \frac{4}{9} \). This can be seen by dividing the numerator and denominator by \( x^3 \)

\[
\lim_{x \to \infty} \frac{4x^3 - 2x}{9x^3 + 1} = \lim_{x \to \infty} \frac{4 - 2/x^2}{9 + 1/x^3}
\]

\[
= \frac{4 - 0}{9 - 0} = \frac{4}{9}
\]
d. \[ \lim_{x \to 0} \frac{4 \sin x}{5x} \]

Recall that \[ \lim_{x \to 0} \frac{\sin x}{x} = 1 \]. Therefore this limit equals \( \frac{4}{5} \).


Consider the limit \( \lim_{x \to 5} (2x - 3) \) which, of course, has the value 7. Since it has the value 7, that means that for each \( \epsilon > 0 \), there exists some \( \delta > 0 \), such that for all \( x \), if \( 0 < |x - 5| < \delta \), then \( |(2x - 3) - 7| < \epsilon \).

Let \( \epsilon = \frac{1}{2} \). Find a value of \( \delta \) that works for this \( \epsilon \).

You need to find a value of \( \delta \) so that

\[ 0 < |x - 5| < \delta \text{ implies } |(2x - 3) - 7| < \frac{1}{2}. \]

The expression \( |(2x - 3) - 7| \) can be rewritten as \( |2x - 10| \) which equals \( 2|x - 5| \). Therefore, the condition \( |(2x - 3) - 7| < \frac{1}{2} \) is equivalent to \( |x - 5| < \frac{1}{4} \).

Thus, you need to find a value of \( \delta \) so that

\[ 0 < |x - 5| < \delta \text{ implies } |x - 5| < \frac{1}{4}. \]

Such a value is \( \delta = \frac{1}{4} \).

6. [10] Suppose that \( \theta \) is an angle between \( -\pi/2 \) and 0, and that \( \cos \theta = \frac{1}{2}\sqrt{2} \). Determine the value of \( \sin \theta \).

Since \( \cos \theta = \frac{1}{2}\sqrt{2} \), the Pythagorean identity \( \sin^2 \theta + \cos^2 \theta = 1 \) implies \( \sin^2 \theta + \frac{1}{2} = 1 \). Hence, \( \sin^2 \theta = \frac{1}{2} \), so \( \sin \theta = \pm \frac{1}{2}\sqrt{2} \). Since \( \theta \) is an angle between \( -\pi/2 \) and 0, the sine of \( \theta \) is negative. Thus \( \sin \theta = -\frac{1}{2}\sqrt{2} \).

7. [15; 5 points each part] Suppose that \( \lim_{x \to \pi} f(x) = 5 \) and \( \lim_{x \to \pi} g(x) = 3 \). Evaluate each of the following limits, or explain why it doesn’t exist.

a. \( \lim_{x \to \pi} \frac{f(x)}{g(x)} \)

Since \( f(x) \) approaches 5, and \( g(x) \) approaches 3, the quotient approaches \( \frac{5}{3} \).