1. [12] On limits of average rates of change. Let \( f(x) = 5x^2 + 4 \).

   a. [4] Write down an expression that gives the average rate of change of this function over the interval between \( x \) and \( x + h \), and simplify the expression.

   \[
   \frac{f(x + h) - f(x)}{h} = \frac{(5(x + h)^2 + 4) - (5x^2 + 4)}{h}
   \]

   b. [8] Compute the limit as \( h \to 0 \) of that average rate of change.

   \[
   \lim_{h \to 0} \frac{(5(x + h)^2 + 4) - (5x^2 + 4)}{h} = \lim_{h \to 0} \frac{5x^2 + 10xh + 5h^2 + 4 - 5x^2 - 4}{h} = \lim_{h \to 0} \frac{10xh + 5h^2}{h} = \lim_{h \to 0} (10x + 5h) = 10x
   \]

2. [10; 5 points each] On the intuitive concept of limit and continuity.

   a. [5] Sketch the graph \( y = f(x) \) of a function for which \( \lim_{x \to 3} f(x) \) does not exist.

   There are many such graphs. For example, if there’s a jump in the value of \( f \) at \( x = 3 \), then that limit won’t exist. See section 2.4 of the text.

   b. [5] Sketch the graph \( y = f(x) \) of a function defined everywhere, the limit \( \lim_{x \to 3} f(x) \) does exist, but \( f \) is not continuous at \( x = 3 \).

   This can be achieved by making \( f(3) \) unequal to the limit, but make sure that the function is defined at \( x = 3 \). See section 2.5 of the text.

3. [10; 5 points each property] On asymptotes.

   a. Sketch the graph of a function \( f \) such that \( \lim_{x \to ^2} f(x) = 0 \) and \( \lim_{x \to ^2} f(x) = -\infty \).

   The graph of the function should should be asymptotic to the vertical line \( x = 2 \). See section 2.6 of the text.

   b. Sketch the graph of a function \( f \) such that \( \lim_{x \to ^-} f(x) = 1 \).

   The graph of the function should should be asymptotic to the horizontal line \( y = 1 \). See section 2.6 of the text.

4. [28; 7 points each part] Evaluate the following limits. If a limit diverges to \( \pm \infty \) it is enough to say that it doesn’t exist.

   a. \( \lim_{x \to 1} \frac{x^2 - 4x + 4}{x^2 - x - 2} \)

   As \( x \) approaches 1, the numerator approaches 1 while the denominator approaches \( -2 \), so the quotient approaches \( -1/2 \).

   b. \( \lim_{x \to ^2} \frac{x^2 - 4x + 4}{x^2 - x - 2} \)

   The expression needs to be simplified before taking the limit.

   \[
   \lim_{x \to ^2} \frac{(x + 2)(x - 2)}{(x + 1)(x - 2)} = \lim_{x \to ^2} \frac{x + 2}{x + 1} = \frac{4}{3}
   \]

   c. \( \lim_{x \to ^\infty} \frac{3x^2 - 2x + 1}{9x^3 + x} \)

   The numerator has a lower degree than the denominator, so as \( x \to \infty \), the limit approaches 0. This can be seen by dividing the numerator and denominator by \( x^3 \)

   \[
   \lim_{x \to ^\infty} \frac{3x^2 - 2x + 1}{9x^3 + x} = \lim_{x \to ^\infty} \frac{3/x - 2/x^2 + 1/x^3}{9 + 1/x^2} = \frac{0 - 0 + 0}{9 + 0} = 0
   \]
d. \( \lim_{x \to 0} \frac{5x}{8 \sin x} \).

Recall that \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \). Therefore \( \lim_{x \to 0} \frac{x}{\sin x} = 1 \), hence this limit equals \( \frac{5}{8} \).


Consider the limit \( \lim_{x \to 3} (9 - 2x) \) which, of course, has the value 3. Since it has the value 3, that means that for each \( \epsilon > 0 \), there exists some \( \delta > 0 \), such that for all \( x \), if \( 0 < |x - 3| < \delta \), then \( |(9 - 2x) - 3| < \epsilon \).

Let \( \epsilon = \frac{1}{3} \). Find a value of \( \delta \) that works for this \( \epsilon \).

You need to find a value of \( \delta \) so that

\[ 0 < |x - 3| < \delta \text{ implies } |(9 - 2x) - 3| < \frac{1}{3}. \]

The expression \( |(9 - 2x) - 3| \) can be rewritten as \( |6 - 2x| \) which equals \( 2|x - 3| \). Therefore, the condition \( |(9 - 2x) - 3| < \frac{1}{3} \) is equivalent to \( |x - 3| < \frac{1}{6} \). Thus, you need to find a value of \( \delta \) so that

\[ 0 < |x - 3| < \delta \text{ implies } |x - 3| < \frac{1}{6}. \]

Such a value is \( \delta = \frac{1}{6} \).

6. [10] Suppose that \( \theta \) is an angle between \( \pi/2 \) and \( \pi \), and that \( \sin \theta = \frac{1}{2} \sqrt{2} \). Determine the value of \( \cos \theta \).

Since \( \sin \theta = \frac{1}{2} \sqrt{2} \), the Pythagorean identity \( \sin^2 \theta + \cos^2 \theta = 1 \) implies \( \frac{1}{2} + \cos^2 \theta = 1 \). Hence, \( \cos^2 \theta = \frac{1}{2} \), so \( \cos \theta = \pm \frac{1}{2} \sqrt{2} \). Since \( \theta \) is an angle between \( \pi/2 \) and \( \pi \), the cosine of \( \theta \) is negative. Thus \( \cos \theta = -\frac{1}{2} \sqrt{2} \).

7. [15; 5 points each part] Suppose that \( \lim_{x \to \pi/2} f(x) = 4 \) and \( \lim_{x \to \pi/2} g(x) = 5 \). Evaluate each of the following limits, or explain why it doesn’t exist.

a. \( \lim_{x \to \pi/2} \frac{f(x) + g(x)}{f(x) - g(x)} \)

Since \( f(x) \) approaches 4, and \( g(x) \) approaches 9, the numerator approaches 9 while the denominator approaches -1. Therefore quotient approaches -9.

b. \( \lim_{x \to \pi/2} \frac{x}{g(x) - f(x) - \sin x} \)

As \( x \) approaches \( \pi/2 \), \( \sin x \) approaches 1. Therefore the denominator approaches 0. But the numerator approaches \( \pi/2 \), so the limit doesn’t exist.

c. \( \lim_{x \to \pi/2} \sqrt{(g(x))^2 + (f(x))^2} \)

The sum \( (g(x))^2 + (f(x))^2 \) approaches 25. Since the square root function is continuous, the limit approaches the square root of 25, which is 5.