The statements of FTC and FTC\(^{-1}\). Before we get to the proofs, let’s first state the Fundamental Theorem of Calculus and the Inverse Fundamental Theorem of Calculus. When we do prove them, we’ll prove FTC\(^{-1}\) before we prove FTC. The FTC is what Oresme propounded back in 1350.

(Sometimes FTC\(^{-1}\) is called the first fundamental theorem and FTC the second fundamental theorem, but that gets the history backwards.)

**Theorem 1** (FTC). If \(F'\) is continuous on \([a, b]\), then

\[
\int_a^b F'(x) \, dx = F(b) - F(a).
\]

In other words, if \(F\) is an antiderivative of \(f\), then

\[
\int_a^b f(x) \, dx = F(b) - F(a).
\]

A common notation for \(F(b) - F(a)\) is \(F(x)\bigg|_a^b\).

There are stronger statements of these theorems that don’t have the continuity assumptions stated here, but these are the ones we’ll prove.

**Theorem 2** (FTC\(^{-1}\)). If \(f\) is a continuous function on the closed interval \([a, b]\), and \(F\) is its accumulation function defined by

\[
F(x) = \int_a^x f(t) \, dt
\]

for \(x\) in \([a, b]\), then \(F\) is differentiable on \([a, b]\) and its derivative is \(f\), that is, \(F'(x) = f(x)\) for \(x \in [a, b]\).

Frequently, the conclusion of this theorem is written

\[
\frac{d}{dx} \int_a^x f(t) \, dt = f(x).
\]

Note that a different variable \(t\) is used in the integrand since \(x\) already has a meaning. Logicians and computer scientists are comfortable using the same variable for two different purposes, but they have to resort to the concept of “scope” of a variable in order to pull that off. It’s usually easier to make sure that each variable only has one meaning. Thus, we use one variable \(x\) as a limit of integration, but a different variable \(t\) inside the integral.

Our first proof is of the FTC\(^{-1}\).
Proof of the FTC\(^{-1}\). First of all, since \(f\) is continuous, it’s integrable, that is to say,

\[
F(x) = \int_a^x f(t) \, dt
\]
does exist.

We need to show that \(F'(x) = f(x)\). By the definition of derivatives,

\[
F'(x) = \lim_{h \to 0} \frac{F(x + h) - F(x)}{h} = \lim_{h \to 0} \frac{1}{h} \left( \int_a^{x+h} f(t) \, dt - \int_a^x f(t) \, dt \right)
\]

We’ll show that this limit equals \(f(x)\). Although a complete proof would consider both cases \(h < 0\) and \(h > 0\), we’ll only look at the case when \(h > 0\); the case for \(h < 0\) is similar but more complicated by negative signs.

We’ll concentrate on the values of the continuous function \(f(x)\) on the closed interval \([x, x+h]\). On this interval, \(f\) takes on a minimum value \(m_h\) and a maximum value \(M_h\) (by the Extremal Value Theorem for continuous functions on closed intervals). Since \(m_h \leq f(t) \leq M_h\) for \(t\) in this interval \([x, x+h]\), therefore when we take the definite integrals on this interval, we have

\[
\int_x^{x+h} m_h \, dt \leq \int_x^{x+h} f(t) \, dt \leq \int_x^{x+h} M_h \, dt.
\]

But \(\int_x^{x+h} m_h \, dt = hm_h\), and \(\int_x^{x+h} M_h \, dt = hM_h\), so, dividing by \(h\), we see that

\[
m_h \leq \frac{1}{h} \int_x^{x+h} f(t) \, dt \leq M_h.
\]

Now, \(f\) is continuous, so as \(h \to 0\) all the values of \(f\) on the shortening interval \([x, x+h]\) approach \(f(x)\), so, in particular, both the minimum value \(m_h\) and the maximum value \(M_h\) approach \(f(x)\). But if both \(m_h\) and \(M_h\) approach the same number \(f(x)\), then anything between them also approaches it, too. Thus

\[
\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f(t) \, dt = f(x)
\]

thereby proving \(F'(x) = f(x)\). Q.E.D.

We’ll now go on to prove the FTC from the FTC\(^{-1}\).

Proof of the FTC. Let

\[
G(x) = \int_a^x F'(t) \, dt.
\]
Then by FTC\(^{-1}\), \(G'(x) = F'(x)\). Therefore, \(G\) and \(F\) differ by a constant \(C\), that is, \(G(x) - F(x) = C\) for all \(x \in [a, b]\). But

\[
G(a) = \int_a^a F'(t) \, dt = 0,
\]

and \(G(a) - F(a) = C\), so \(C = -F(a)\). Hence, \(G(x) - F(x) = -F(a)\) for all \(x \in [a, b]\). In particular, \(G(b) - F(b) = -F(a)\), so \(G(b) = F(b) - F(a)\), that is,

\[
\int_a^b F'(t) \, dt = F(b) - F(a).
\]

Q.E.D.

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