Math 130 Linear Algebra
Final Exam
December 2012

You may use a calculator and a sheet of notes. Leave your answers as expressions such as $e^2 \sqrt{\frac{\sin^2(\pi/6)}{1 + \ln 10}}$ if you like. Show all your work for credit. Points for each problem are in square brackets.

1. [12] Consider the three vectors $u = (1, 3, 1)$, $v = (4, 2, -1)$, and $w = (-3, 1, 2)$.
   
   a. [8] Either prove that the $u$ is in the span of the vectors $v$ and $w$, or prove that it is not. (There are several ways you can approach this question. Any one will do.)

   b. [4] Are the three vectors $u$, $v$, and $w$ linearly dependent, or linearly independent?
2. [16] Let $A$ be the matrix 

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

a. [4] Write down the characteristic polynomial $f(\lambda)$ for $A$.


c. [8] For each of the eigenvalues of $A$, find the eigenspace of eigenvectors for that eigenvalue.
3. [15; 5 points each part] On dimension.
   
a. State the definition of the dimension of a vector space.

b. What theorem on vector spaces is required before this definition of dimension is a valid definition?

c. Let $V$ be the vector space $V = \{(w, x, y, z) \in \mathbb{R}^4 \mid w = x + y + z\}$. Give an example of a 2-dimensional subspace $W$ of $V$. 

4. [15] If $A$ is a $5 \times 3$ matrix, show that the rows of $A$ are linearly dependent. (There are several ways that you can approach this. Be sure to write a clear and complete explanation.)

5. [15] Suppose that $u$, $v$, and $w$ are vectors in $\mathbb{R}^3$. Prove that if $u$ is orthogonal to both $v$ and $w$, then $u$ is also orthogonal to $8v + 13w$. 
6. [20; 4 points each part] Consider the three vectors in the plane \( \mathbf{u} = (2, 1) \), \( \mathbf{v} = (2, 3) \), and \( \mathbf{w} = (6, 1) \).

a. Compute \( ||\mathbf{v} - \mathbf{w}|| \).

b. Compute the inner product \( \langle \mathbf{u} | \mathbf{v} \rangle \).

c. Let \( \theta \) be the angle between \( \mathbf{u} \) and \( \mathbf{v} \). Find \( \cos \theta \).

d. Which of the vectors \( \mathbf{u} \), \( \mathbf{v} \), and \( \mathbf{w} \) are parallel, if any?

e. Which of the vectors \( \mathbf{u} \), \( \mathbf{v} \), and \( \mathbf{w} \) are orthogonal, if any?
7. [10] Evaluate the following determinant using any method you like. Show all your work.

\[
|A| = \begin{vmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & 1 \\ 5 & 3 & 0 & 0 \\ 0 & 0 & 2 & -1 \end{vmatrix}
\]