1. [12] Consider the three vectors \( u = (1, 3, 1), \ v = (4, 2, -1), \) and \( w = (-3, 1, 2). \)

   a. [8] Either prove that the \( u \) is in the span of the vectors \( v \) and \( w, \) or prove that it is not. (There are several ways you can approach this question. Any one will do.)
   
   Note that \( u = v + w. \) Since \( u \) is a linear combination of \( v \) and \( w, \) it’s in the span of them.

   b. [4] Are the three vectors \( u, \ v, \) and \( w \) linearly dependent, or linearly independent?
   
   They’re linearly dependent since one of them is a linear combination of the others.

2. [16] Let \( A \) be the matrix \( A = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}. \)

   a. [4] Write down the characteristic polynomial \( f(\lambda) \) for \( A. \)

   The determinant of the matrix
   
   \[
   A - \lambda I = \begin{bmatrix} 2 - \lambda & 2 & 2 \\ 0 & 2 - \lambda & 0 \\ 0 & 0 & 3 - \lambda \end{bmatrix}
   \]

   is the characteristic polynomial. Since that’s an upper triangular matrix, its determinant is the product of its diagonal entries, \((2 - \lambda)^2(3 - \lambda).\)

   b. [4] Determine the eigenvalues for \( A. \)

   The eigenvalues are the roots of the characteristic polynomial, namely, 2 and 3. (Note that 2 has multiplicity 2.)

   c. [8] For each of the eigenvalues of \( A, \) find the eigenspace of eigenvectors for that eigenvalue.

   Case \( \lambda = 2. \) The solutions to the matrix equation \((A - 2I)x = 0\) are \( x = (x, 0, 0) \) where \( x \) is an arbitrary number. (Note that even though 2 has multiplicity 2, the dimension of its eigenspace is only 1.)

   Case \( \lambda = 3. \) The solutions to the matrix equation \((A - 3I)x = 0\) are \( x = (2z, 0, z) \) where \( z \) is an arbitrary number.

3. [15; 5 points each part] On dimension.

   a. State the definition of the dimension of a vector space.

   The dimension of a vector space is the number of vectors in any of its bases.

   b. What theorem on vector spaces is required before this definition of dimension is a valid definition?

   The theorem that says that every basis of a vector space has the same number of vectors. That followed directly from the replacement theorem.

   c. Let \( V \) be the vector space \( V = \{(w, x, y, z) \in \mathbb{R}^4 | w = x + y + z\}. \) Give an example of a 2-dimensional subspace \( W \) of \( V. \)

   As \( V \) has dimension 3, you can require almost any equation to reduce it to dimension 2. For example, if you require \( x = 0, \) then you get the 2-dimensional subspace \( W = \{(w, 0, y, z) \in \mathbb{R}^4 | w = y + z\}. \)

4. [15] If \( A \) is a \( 5 \times 3 \) matrix, show that the rows of \( A \) are linearly dependent. (There are several ways that you can approach this. Be sure to write a clear and complete explanation.)

   One such proof: The column space is spanned by the three columns, so it has at most dimension 3. Therefore the rank of \( A \) is at most 3. Therefore, the row space has at most dimension 3. Since there are 5 rows, they can’t be independent since the row space can’t be as high as 5.

5. [15] Suppose that \( u, \ v, \) and \( w \) are vectors in \( \mathbb{R}^3. \) Prove that if \( u \) is orthogonal to both \( v \) and \( w, \) then \( u \) is also orthogonal to \( 8v + 13w. \)
Since \( \mathbf{u} \perp \mathbf{v} \) and \( \mathbf{u} \perp \mathbf{w} \), therefore \( \langle \mathbf{u} | \mathbf{v} \rangle = 0 \) and \( \langle \mathbf{u} | \mathbf{w} \rangle = 0 \). Thus,
\[
\langle \mathbf{u} | 8\mathbf{v} + 13\mathbf{w} \rangle = \langle \mathbf{u} | 8\mathbf{v} \rangle + \langle \mathbf{u} | 13\mathbf{w} \rangle
= 8\langle \mathbf{u} | \mathbf{v} \rangle + 13\langle \mathbf{u} | \mathbf{w} \rangle
= 8 \cdot 0 + 13 \cdot 0 = 0
\]
Hence, \( \mathbf{u} \perp (8\mathbf{v} + 13\mathbf{w}) \). Q.E.D.

6. [20; 4 points each part] Consider the three vectors in the plane \( \mathbf{u} = (2, 1), \mathbf{v} = (2, 3), \) and \( \mathbf{w} = (6, 1) \).

a. Compute \( \| \mathbf{v} - \mathbf{w} \| \).
\[
\| (2, 3) - (6, 1) \| = \| (-4, 2) \| = \sqrt{16 + 4} = \sqrt{20}
\]
b. Compute the inner product \( \langle \mathbf{u} | \mathbf{v} \rangle \).
\[
\langle (2, 1) | (2, 3) \rangle = 2 \cdot 2 + 1 \cdot 3 = 7
\]
c. Let \( \theta \) be the angle between \( \mathbf{u} \) and \( \mathbf{v} \). Find \( \cos \theta \).
\[
\cos \theta = \frac{\langle \mathbf{u} | \mathbf{v} \rangle}{\| \mathbf{u} \| \| \mathbf{v} \|} = \frac{7}{\sqrt{5} \sqrt{13}} = \frac{7}{\sqrt{65}}
\]
d. Which of the vectors \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) are parallel, if any?
None are parallel since no one of them is a scalar multiple of any other one.

e. Which of the vectors \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) are orthogonal, if any?
None are orthogonal since the inner products of any two of them are nonzero.

7. [10] Evaluate the following determinant using any method you like. Show all your work.
\[
| \mathbf{A} | = \begin{vmatrix}
1 & 0 & -3 & 0 \\
0 & 1 & 0 & 1 \\
5 & 3 & 0 & 0 \\
0 & 0 & 2 & -1
\end{vmatrix}
\]
There are many ways to compute the determinant. It is \(-9\).