CLARK UNIVERSITY 1887

Final Exam Answers
Math 130 Linear Algebra

D Joyce, December 2012

Scale. 86–103 A, 70–85 B, 55–69 C. Median 75.

1. [12] Consider the three vectors $\mathbf{u} = (1, 3, 1)$, $\mathbf{v} = (4, 2, -1)$, and $\mathbf{w} = (-3, 1, 2)$.

a. [8] Either prove that the $\mathbf{u}$ is in the span of the vectors $\mathbf{v}$ and $\mathbf{w}$, or prove that it is not. (There are several ways you can approach this question. Any one will do.)

Note that $\mathbf{u} = \mathbf{v} + \mathbf{w}$. Since $\mathbf{u}$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$, it's in the span of them.

b. [4] Are the three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ linearly dependent, or linearly independent?

They're linearly dependent since one of them is a linear combination of the others.

2. [16] Let A be the matrix $A = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

a. [4] Write down the characteristic polynomial $f(\lambda)$ for A.

The determinant of the matrix

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 2 & 2 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}$$

is the characteristic polynomial. Since that's an upper triangular matrix, its determinant is the product of its diagonal entries, $(2-\lambda)^2(3-\lambda)$.

b. [4] Determine the eigenvalues for A.

The eigenvalues are the roots of the characteristic polynomial, namely, 2 and 3. (Note that 2 has multiplicity 2.)

c. [8] For each of the eigenvalues of A, find the eigenspace of eigenvectors for that eigenvalue.

Case $\lambda = 2$. The solutions to the matrix equation $(A - 2I)\mathbf{x} = \mathbf{0}$ are $\mathbf{x} = (x, 0, 0)$ where x is an arbitrary number. (Note that even though 2 has multiplicity 2, the dimension of its eigenspace is only 1.)

Case $\lambda = 3$. The solutions to the matrix equation $(A - 3I)\mathbf{x} = \mathbf{0}$ are $\mathbf{x} = (2z, 0, z)$ where z is an arbitrary number

3. [15; 5 points each part] On dimension.

a. State the definition of the dimension of a vector space.

The dimension of a vector space is the number of vectors in any of its bases.

b. What theorem on vector spaces is required before this definition of dimension is a valid definition?

The theorem that says that every basis of a vector space has the same number of vectors. That followed directly from the replacement theorem.

c. Let V be the vector space $V = \{(w, x, y, z) \in \mathbf{R}^4 \mid w = x + y + z\}$. Give an example of a 2-dimensional subspace W of V.

As V has dimension 3, you can require almost any equation to reduce it to dimension 2. For example, if you require $x = 0$, then you get the 2-dimensional subspace $W = \{(w, 0, y, z) \in \mathbf{R}^4 \mid w = y + z\}$.

4. [15] If A is a 5×3 matrix, show that the rows of A are linearly dependent. (There are several ways that you can approach this. Be sure to write a clear and complete explanation.)

One such proof: The column space is spanned by the three columns, so it has at most dimension 3. Therefore the rank of A is at most 3. Therefore, the row space has at most dimension 3. Since there are 5 rows, they can't be independent since the row space can't be as high as 5.

5. [15] Suppose that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbf{R}^3$. Prove that if $\mathbf{u}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, then $\mathbf{u}$ is also orthogonal to $8\mathbf{v} + 13\mathbf{w}$.

Since $\mathbf{u} \perp \mathbf{v}$ and $\mathbf{u} \perp \mathbf{w}$, therefore $\langle \mathbf{u}|\mathbf{v}\rangle = 0$ and $\langle \mathbf{u}|\mathbf{w}\rangle = 0$. Thus,

$$\begin{aligned}\langle \mathbf{u}|8\mathbf{v} + 13\mathbf{w}\rangle &= \langle \mathbf{u}|8\mathbf{v}\rangle + \langle \mathbf{u}|13\mathbf{w}\rangle \\ &= 8\langle \mathbf{u}|\mathbf{v}\rangle + 13\langle \mathbf{u}|\mathbf{w}\rangle \\ &= 8 \cdot 0 + 13 \cdot 0 = 0\end{aligned}$$

Hence, $\mathbf{u} \perp (\mathbf{8v} + \mathbf{13w})$. Q.E.D.

6. [20; 4 points each part] Consider the three vectors in the plane $\mathbf{u} = (2, 1)$, $\mathbf{v} = (2, 3)$, and $\mathbf{w} = (6, 1)$.

a. Compute $\|\mathbf{v} - \mathbf{w}\|$.

$$\|(2, 3) - (6, 1)\| = \|(-4, 2)\| = \sqrt{16 + 4} = \sqrt{20}$$

b. Compute the inner product $\langle \mathbf{u}|\mathbf{v}\rangle$.

$$\langle (2, 1)|(2, 3)\rangle = 2 \cdot 21 \cdot 3 = 7$$

c. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Find $\cos\theta$.

$$\cos\theta = \frac{\langle \mathbf{u}|\mathbf{v}\rangle}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{7}{\sqrt{5}\sqrt{13}} = \frac{7}{\sqrt{65}}$$

d. Which of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are parallel, if any?

None are parallel since no one of them is a scalar multiple of any other one.

e. Which of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are orthogonal, if any?

None are orthogonal since the inner products of any two of them are nonzero.

7. [10] Evaluate the following determinant using any method you like. Show all your work.

$$|A| = \begin{vmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & 1 \\ 5 & 3 & 0 & 0 \\ 0 & 0 & 2 & -1 \end{vmatrix}$$

There are many ways to compute the determinant. It is -9.