CLARK UNIVERSITY 1887

First Test Answers
Math 130 Linear Algebra
D Joyce, October 2012

Scale. 86–100 A, 71–85 B, 50–70 C. Median 88 (A-).

1. [10] Name two subspaces of $\mathbf{R}^2$ whose union is not a subspace of $\mathbf{R}^2$. (No need to give a proof.)

There aren't many subspaces of $\mathbf{R}^2$. There's the trivial subspace that consists of the origin $\mathbf{0}$ alone, and the the entire plane $\mathbf{R}^2$, but those won't help in giving and answer here.

The rest of the subspaces of $\mathbf{R}^2$ are the straight lines through $\mathbf{0}$. Any two of these will do. A line through the origin is a subspace of the plane, yet the union of two lines is not a subspace. The reason is that if you add a vector on one line to a vector on the other, their sum will not be on either line. That means the union is not a subspace.

2. [12] Explain in a sentence or two why it is the case that if S is a linearly independent set of vectors in a vector space, then any subset S' of S is also a linearly independent set of vectors in that vector space.

A set of vectors is independent if the only linear combination of them that is $\mathbf{0}$ is the trivial linear combination. If that's true for linear combinations in a set S, it will automatically be true for any linear combination in a subset S' of S since a linear combination of S' is a linear combination of S.

An explanation will have to describe what it means for a set of vectors to be linearly independent (or, if you prefer, what it means for them to be dependent). Without the connection to the meaning of the concept, an explanation would be incomplete.

3. [25] Consider the system of linear equations

$$\begin{array}{rcrcrcrcr} x & + & y & + & 3z & + & 2w & = & 7 \\ 2x & - & y & & & + & 4w & = & 8 \\ & & 3y & + & 6z & & & = & 6 \end{array}$$

a. [5] Write down the augmented matrix for this system.

$$\begin{bmatrix} 1 & 1 & 3 & 2 & 7 \\ 2 & -1 & 0 & 4 & 8 \\ 0 & 3 & 6 & 0 & 6 \end{bmatrix}$$

b. [10] Use row operations to convert this matrix to echelon or reduced echelon form. (Your choice, but show your work.)

$$\begin{bmatrix} 1 & 1 & 3 & 2 & 7 \\ 0 & -3 & -6 & 0 & -6 \\ 0 & 3 & 6 & 0 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 3 & 2 & 7 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

c. [10] Describe all the solutions for the original system of linear equations.

One way to describe the solutions is to say the general solution is

$$(x, y, z, w) = (5 - z - 2w, 2 - 2z, z, w)$$

where z and w can be any scalars.

You could also write the general solution as

$$(x, y, z, w) = (5, 2, 0, 0) + z(-1, -2, 1, 0) + w(-1, 0, 0, 1)$$

where, again, z and w can be any scalars.

You could also parametrize the solutions in terms of x and y instead of z and w, or, for that matter, any two of the four variables x, y, z, and z.

4. [25] Using only the axioms of vector spaces pertaining to addition prove the law of cancellation for addition:

$$\mathbf{u} + \mathbf{v} = \mathbf{u} + \mathbf{w} \text{ implies } \mathbf{v} = \mathbf{w}.$$

The axioms of vector spaces pertaining to addition are

1. Vector addition is commutative: $\mathbf{v}+\mathbf{w} = \mathbf{w}+\mathbf{v}$ for all vectors $\mathbf{v}$ and $\mathbf{w}$;

2. Vector addition is associative: $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$ for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$;

3. There is a vector, denoted $\mathbf{0}$ and called the *zero vector*, such that $\mathbf{v}+\mathbf{0} = \mathbf{v} = \mathbf{0}+\mathbf{v}$ for each vector $\mathbf{v}$;

4. For each vector $\mathbf{v}$, there is another vector denoted $(-\mathbf{v})$ such that $\mathbf{v}+(-\mathbf{v}) = \mathbf{0}$.

In your proof, do not use the operation of subtraction, but use the fourth axiom as it is stated.

You can write your proof in several different forms. I'll give a narrative proof, a two-column proof, and an equational proof.

The proofs I have here aren't the only ones. There are different orders you can use the axioms, and the equations don't have to be the ones shown here.

Narrative proof. Suppose that

$$\mathbf{u}+\mathbf{v} = \mathbf{u}+\mathbf{w}.$$

By axiom 3, there is a vector $-\mathbf{u}$. Add it to both sides of the equation on the left to get

$$(-\mathbf{u})+(\mathbf{u}+\mathbf{v}) = (-\mathbf{u})+(\mathbf{u}+\mathbf{w}).$$

By associatively, axiom 2, we can change the placement of parentheses to get

$$((-\mathbf{u})+\mathbf{u})+\mathbf{v} = ((-\mathbf{u})+\mathbf{u})+\mathbf{w}.$$

By commutativity, axiom 1, we can exchange $-\mathbf{u}$ and $\mathbf{u}$ to get

$$(\mathbf{u}+(-\mathbf{u}))+\mathbf{v} = (\mathbf{u}+(-\mathbf{u}))+\mathbf{w}.$$

But axiom 3 says $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$, so that equation simplifies to

$$\mathbf{0}+\mathbf{v} = \mathbf{0}+\mathbf{w},$$

which by axiom 4 further simplifies to

$$\mathbf{v} = \mathbf{w}.$$

Thus, we have shown that $\mathbf{u}+\mathbf{v} = \mathbf{u}+\mathbf{w}$ implies $\mathbf{v} = \mathbf{w}$. Q.E.D.

Two-column proof. You give the assertions in the left column and abbreviated reasons in the right column. These require the reader interpret what you say. In this particular proof, each follows from the previous, but in more complicated proofs a line may follow from more than one of the previous lines, and for those, you should state which lines are used. Also this is a proof of a simple implication of the form $P \implies Q$, so it's enough to state P on the first line and end with Q.

1.	$\mathbf{u}+\mathbf{v} = \mathbf{u}+\mathbf{w}$	Given
2.	$(-\mathbf{u})+(\mathbf{u}+\mathbf{v}) = (-\mathbf{u})+(\mathbf{u}+\mathbf{w})$	Ax 3
3.	$((-\mathbf{u})+\mathbf{u})+\mathbf{v} = ((-\mathbf{u})+\mathbf{u})+\mathbf{w}$	Ax 2
4.	$(\mathbf{u}+(-\mathbf{u}))+\mathbf{v} = (\mathbf{u}+(-\mathbf{u}))+\mathbf{w}$	Ax 1
5.	$\mathbf{0}+\mathbf{v} = \mathbf{0}+\mathbf{w}$	Ax 3
6.	$\mathbf{v} = \mathbf{w}$	Ax 4

Equational proof. Some proofs, like this one, can be written as one long, continued equation. They're easy to check, but they usually have to be constructed from other proofs. When you read one you're often puzzled as to where it came from. Here's what this proof looks like as an equation where each equality has an associated justification.

$$\begin{aligned}
\mathbf{v} &= \mathbf{0}+\mathbf{v} && \text{by axiom 4}\\
&= (\mathbf{u}+(-\mathbf{u}))+\mathbf{v} && \text{by axiom 3}\\
&= ((-\mathbf{u})+\mathbf{u})+\mathbf{v} && \text{by axiom 1}\\
&= (-\mathbf{u})+(\mathbf{u}+\mathbf{v}) && \text{by axiom 2}\\
&= (-\mathbf{u})+(\mathbf{u}+\mathbf{w}) && \text{given}\\
&= ((-\mathbf{u})+\mathbf{u})+\mathbf{w} && \text{by axiom 2}\\
&= (\mathbf{u}+(-\mathbf{u}))+\mathbf{w} && \text{by axiom 1}\\
&= \mathbf{0}+\mathbf{w} && \text{by axiom 3}\\
&= \mathbf{w} && \text{by axiom 4}
\end{aligned}$$

5. [28; 4 points each] True/false.

a. The solutions to a system of linear equations form a vector space. *False.* That's only true when the linear equations are homogeneous.

b. Every subspace of a vector space must contain the zero vector, $\mathbf{0}$. *True.* Vector spaces have to have $\mathbf{0}$.

c. The set of all polynomials with real coefficients forms a vector space over the field $\mathbf{R}$. *True.* The vector space is infinite dimensional, but it is a vector space.

d. Every symmetric matrix is an upper triangular matrix. *False.* The only symmetric matrices that are upper triangular matrices are the diagonal matrices.

e. If two nonzero vectors are linearly dependent, then one is a scalar multiple of the other. *True.* However, three nonzero vectors can be linearly dependent without any of them being scalar multiples of any of the others.

f. If the dimension of a vector space V is m, and the dimension of a vector space W is n, then the dimension of the product space $V \times W$ is mn. *False.* The dimension of the product is $m + n$. For example, $\mathbf{R}^2 \times \mathbf{R}^1$ is isomorphic to $\mathbf{R}^3$, and it's dimension is 3, not 2.

g. If there are n independent vectors in a vector space, then every spanning set for that vector space has at least n vectors. *True.* The n independent vectors cannot be all be linear combinations of fewer than n vectors.