CLARK UNIVERSITY 1887

Assignment 2 answers
Math 130 Linear Algebra
D Joyce, Fall 2013

Exercises from section 1.2, page 12, exercises 1, 4a-d, 11, 12, 18, 22. See the answers in the book for exercise 4.

1. True/false.

a. Every vector space contains a zero vector.

True. The existence of $\mathbf{0}$ is a requirement in the definition.

b. A vector space may have more than one zero vector.

False. That's not an axiom, but you can prove it from the axioms. Suppose that $\mathbf{z}$ acts like a zero vector, that is to say, $\mathbf{v}+\mathbf{z}=\mathbf{v}$ for every vector $\mathbf{v}$. Then in particular, $\mathbf{0}+\mathbf{z}=\mathbf{0}$. But $\mathbf{z}=\mathbf{0}+\mathbf{z}$. Therefore, $\mathbf{z}=\mathbf{0}$. Thus there can be only one vector with the properties of a zero vector.

c. In any vector space $a\mathbf{x}=b\mathbf{x}$ implies that $a=b$.

False. It's only true if you add the hypothesis that $\mathbf{x}\neq\mathbf{0}$.

d. In any vector space $a\mathbf{x}=a\mathbf{y}$ implies that $\mathbf{x}=\mathbf{y}$.

False. It's only true if you add the hypothesis that $a\neq 0$.

e. A vector in F^n may be regarded as a matrix in $M_{n\times 1}(F)$.

True. We we'll frequently regard vectors as row matrices. We'll also regard them as column matrices. They're not exactly the same thing, but F^n is isomorphic to $M_{n\times 1}(F)$ and to $M_{1\times n}(F)$, so for many purposes we can identify them.

f. An $m\times n$ matrix had m columns and n rows.

False. It's the other way around.

g. In $P(F)$ only polynomials of the same degree can be added.

False. You can add any polynomials.

h. If f and g are polynomials of degree n, then $f+g$ is also a polynomial of degree n.

False. The sum will have a lower degree if the leading coefficient of g is the negation of the leading coefficient of f.

i. If f is a polynomial of degree n and c is a nonzero scalar, then cf is a polynomial of degree n.

True.

j. A nonzero scalar of F may be considered to be a polynomial in $P(F)$ having degree zero.

True.

k. Two functions in $\mathcal{F}(S,F)$ are equal if and only if they have the same value at each element of S.

True. They can be named by different formulas, but if they act the same, then they're the same function.

11. Let $V=\{\mathbf{0}\}$ consist of a single vector $\mathbf{0}$, and define $\mathbf{0}+\mathbf{0}=\mathbf{0}$ and $c\mathbf{0}=\mathbf{0}$ for each scalar c in the field F. Prove that V is a vector space over F. (V is called the *zero vector space.*)

It's a set with the two operations. We just have to check each of the 8 axioms.

VS1. Commutativity of addition. Yes, $\mathbf{x}+\mathbf{y}$ does equal $\mathbf{y}+\mathbf{x}$ because both are $\mathbf{0}$, the only vector in the space.

The same argument applies to verify VS2 and VS5 through VS8. Each axiom is an equation which says one vector equals another vector, but since there's only one vector, namely $\mathbf{0}$, both sides have to be that vector.

Axiom VS3 is a little different in that it says that there is a vector $\mathbf{0}$ with a certain property, but there is such a vector in V, namely $\mathbf{0}$. Likewise axiom VS4 says for a vector $\mathbf{x}$ there is a vector $\mathbf{y}$ so that $\mathbf{x}+\mathbf{y}=\mathbf{0}$. Take the $\mathbf{y}$ to be $\mathbf{0}$ and that equation holds (since all vectors equal $\mathbf{0}$).

Thus, it's a vector space over F. Q.E.D.

12. A function $f:\mathbf{R}\to\mathbf{R}$ is an *even function* if $f(-t)=f(t)$ for each real number t. Prove that

the set $\mathcal{E}$ of even functions with the operations of addition and scalar multiplication defined in the text for functions is a vector space.

First, we need to show that $\mathcal{E}$ has the operations of a vector space.

Yes, $+$ is defined on all functions $f : \mathbf{R} \to \mathbf{R}$, but is it defined on $\mathcal{E}$? If f and g are two even functions, then $f + g$ is a function $\mathbf{R} \to \mathbf{R}$, but is it in $\mathcal{E}$? We'll show that it is.

So, let f and g be even functions. Then $(f + g)(-t)$ is defined to be $f(-t)+g(-t)$, which, since f and g are even functions, is equal to $f(t)+g(t)$, and that, in turn, equals $(f+g)(t)$. Thus, $(f+g)(-t) = (f + g)(t)$ for all t, so $f + g$ is an even function. In other words, $\mathcal{E}$ is closed under addition.

We also have to show for $f \in \mathcal{E}$ and $c \in \mathbf{R}$ that $cf \in \mathcal{E}$. But

$$(cf)(-t) = cf(-t) = cf(t) = (cf)(t).$$

Therefore $\mathcal{E}$ is closed under scalar multiplication.

Next, we need to show that all 8 axioms hold. All the equations hold for functions $\mathbf{R} \to \mathbf{R}$, so they still hold when the functions happen to be even. The only axioms to check are axioms VS3 and VS4 which say certain vectors exist.

For VS3, you need to show that $\mathbf{0} \in \mathcal{E}$. The function $\mathbf{0}$ is the zero function defined by $\mathbf{0}(t) = 0$. It's even since $\mathbf{0}(-t)$ also equals 0.

For VS4, let f be an even function. Show that $-f$ is also an even function.

$$(-f)(-x) = -f(-x) = -f(x) = (-f)(x).$$

All the axioms are satisfied, so $\mathcal{E}$ is a vector space.

(What we did here we'll soon generalize. If a subset of a vector space has $\mathbf{0}$ and is closed under addition and scalar multiplication then it's a vector space itself, a subspace of the original vector space.)

18. Define addition and scalar multiplication on $V = \mathbf{R}^2$ by

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2)$$

and

$$c(a_1, a_2) = (ca_1, ca_2).$$

Is V a vector space over $\mathbf{R}$ with these operations?

The operations are well defined, so it's a matter of checking each of the 8 axioms. If all of them hold, it's a vector space. If even one axiom doesn't hold, then it's not a vector space.

Let's start with VS1. Is addition commutative? Does $(a_1, a_2)+(b_1, b_2)$ equal $(b_1, b_2)+(a_1, a_2)$? That is, is it true that

$$(a_1 + 2b_1, a_2 + 3b_2) = (b_1 + 2a_1, b_2 + 3a_2)$$

for all values of the variables? Look at the first coordinates first. Does a_1+2b_1 equal b_1+2a_1 for all values of a_1 and b_1? No. For instance if $a_1 = 1$ and $b_1 = 0$, it is not the case that 1 equals 2. Therefore, addition is not commutative. Since VS1 fails, this is not a vector space.

(You could, of course, have found that some other axiom doesn't hold like VS7. Once you found that any one axiom doesn't hold, though, that's enough to answer the question.)

22. How many matrices are there in the vector space $M_{m\times n}(\mathbf{Z}_2)$?

The question is, how many different ways and you fill in an $m \times n$ matrix with 0's and 1's?

For example, if you have a 2×3 matrix, you have 6 entries, three in the first row and three in the second. Each one of those 6 entries can be 0 or 1. So the total number of ways to fill in the matrix is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$, which is 2^6.

There are mn entries in an $m \times n$ matrix, and each can be 0 or 1, so that makes 2^{mn} matrices.

Math 130 Home Page at
`http://math.clarku.edu/~djoyce/ma130/`