CLARK UNIVERSITY 1887

Assignment 4 answers
Math 130 Linear Algebra
D Joyce, Fall 2013

Exercises from section 1.4, page 32, exercises 1, 2ad, 3ad, 5de, 6, 9, 14.

1. True/false

a. The zero vector is a linear combination of any nonempty set of vectors.

True. It's $\mathbf{0} = 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_n$. Moreover, an empty sum, that is, the sum of no vectors, is usually defined to be $\mathbf{0}$, and with that definition $\mathbf{0}$ is a linear combination of any set of vectors, empty or not.

b. The span of the empty set $\emptyset$ is $\emptyset$.

False. It's the trivial vector space that includes only the zero vector $\mathbf{0}$. For convenience, our text made that a special case of the definition. If you define an empty sum to be $\mathbf{0}$, then you don't have to make it a special case.

c. If S is a subset of a vector space V, then $\operatorname{span}(S)$ is the intersection of all subspaces of V containing S.

True. See theorem 5.

d. In solving a system of linear equations, it is permissible to multiply an equation by any constant.

False. It's almost true, but you can't multiply it by 0. That would be the same as erasing the equation.

e. In solving a system of linear equations, it is permissible to add any multiple of one equation to another.

True. (But you can't add -1 times the equation to itself.)

f. Every system of linear equations has a solution.

False. There are inconsistent systems like $x-y=3$ and $x-y=4$. Note, however, that every system of homogeneous equations has a solution, namely $\mathbf{0}$.

2d. Solve the system of linear equations.

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & + & 2x_3 & & & = & 2 \\ x_1 & & & + & 8x_3 & + & 5x_4 & = & -6 \\ x_1 & + & x_2 & + & 5x_2 & + & 5x_4 & = & 3 \end{array}$$

You can solve this by hand in many ways using row operations. Here's one. You'd probably start by subtracting the first from each of the second and third rows to get

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & + & 2x_3 & & & = & 2 \\ & - & 2x_2 & + & 6x_3 & + & 5x_4 & = & -8 \\ & - & x_2 & + & 3x_3 & + & 5x_4 & = & 1 \end{array}$$

At this point there's more than one natural way to continue. Since I don't like to work with fractions, I'll switch the second and third rows, then negate the new second row to get

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & + & 2x_3 & & & = & 2 \\ & & x_2 & - & 3x_3 & - & 5x_4 & = & -1 \\ & - & 2x_2 & + & 6x_3 & + & 5x_4 & = & -8 \end{array}$$

Next, I'll add twice the second row to the third row to get

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & + & 2x_3 & & & = & 2 \\ & & x_2 & - & 3x_3 & - & 5x_4 & = & -1 \\ & & & & & - & 5x_4 & = & -10 \end{array}$$

And divide the third row by -5.

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & + & 2x_3 & & & = & 2 \\ & & x_2 & - & 3x_3 & - & 5x_4 & = & -1 \\ & & & & & & x_4 & = & 2 \end{array}$$

The system is now in echelon form. I'm going to continue to put it into reduced echelon form. First I'll add 5 times the third row to the second.

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & + & 2x_3 & & & = & 2 \\ & & x_2 & - & 3x_3 & + & & = & 9 \\ & & & & & & x_4 & = & 2 \end{array}$$

Then subtract twice the second from the first.

$$\begin{array}{rcrcrcrcr} x_1 & + & & + & 8x_3 & & & = & -16 \\ & & x_2 & + & -3x_3 & + & & = & 9 \\ & & & & & & x_4 & = & 2 \end{array}$$

It's in reduced echelon form. The third equation tells me that $x_4 = 2$; the second that $x_2 = 3x_3 + 9$; and the first that $x_1 = -8x_3 - 16$. None of them say what x_3 is, and that's because x_3 can be any number. Thus, the general solution is

$$(x_1, x_2, x_3, x_4) = (-8x_3 - 16, 3x_3 + 9, x_3, 2)$$

where x_3 can be any number. We've found the general solution parameterized by the variable x_3.

You may have done it differently and found the general solution parameterized some other way.

All solutions will describe the same set in $\mathbf{R}^4$. It's a particular straight line that doesn't pass through the origin. That's easier to see if you write the solutions as

$$\mathbf{x} = t(-8, 3, 1, 0) + (-16, 9, 0, 2)$$

where $t = x_3$ is a parameter.

3. Is the first vector a linear combination of the other two?

a. $(-2, 0, 3), (1, 3, 0), (2, 4, -1)$

The general method to solve questions like this is to solve a vector equation, in this case, the vector equation

$$x(1, 3, 0) + y(2, 4, -1) = (-2, 0, 3).$$

That corresponds to a system of three equations in two unknowns

$$\begin{array}{rcrcr} x & + & 2y & = & -2 \\ 3x & + & 4y & = & 0 \\ & - & y & = & 3 \end{array}$$

Then use whatever method you want to solve the system. This particular system has a solution, $(x, y) = (4, -3)$. Therefore, the first vector is a linear combination of the other two.

d. $(2, -1, 0), (1, 2, -3), (1, -3, 2)$

Solve the vector equation

$$x(1, 2, -3) + y(1, -3, 2) = (1, -1, 0)$$

by solving the system of linear equations

$$\begin{array}{rcrcr} x & + & y & = & 1 \\ 2x & - & 3y & = & -1 \\ -3x & + & 2y & = & 0 \end{array}$$

which has the solution $(x, y) = (\frac{2}{5}, \frac{3}{5})$, so the first vector is a lin combo of the other two.

5. Determine if the given vector $\mathbf{v}$ is in the span of a set S.

d. $\mathbf{v} = (2, -1, 1, -3)$,
$S = \{(1, 0, 1, -1), (0, 1, 1, 1)\}$.

This is the same question as exercise 3, but worded differently, so you can answer it in the same way, namely, by solving the vector equation

$$x(1, 0, 1, -1) + y(0, 1, 1, 1) = (2, -1, 1, -3).$$

This one is particularly easy to solve. x has to be 2 for the first coordinate to work out, and y has to be -1 for the second coordinate. And with those values of x and y, the third and fourth coordinates match, too. So there is a solution, and that means $\mathbf{v}$ is in the span of S.

e. $\mathbf{v} = -x^3 + 2x^2 + 3x + 3$,
$S = \{x^3 + x^2 + x + 1, x^2 + x + 1, x + 1\}$.

Here, the vector space is the set of polynomials of degree 3 or less. There are various shortcuts, but you could use the same method of solving the vector equation

$$s(x^3 + x^2 + x + 1) + t(x^2 + x + 1) + u(x + 1) = -x^3 + 2x^2 + 3x + 3.$$

That corresponds to the system of linear equations (one equation for each power of x)

$$\begin{array}{rcrcrcr} s & & & & & = & -1 \\ s & + & t & & & = & 2 \\ s & + & t & + & u & = & 3 \\ s & + & t & + & u & = & 3 \end{array}$$

Since there's a solution, namely, $(s, t, u) = (1, 1, 1)$, therefore $\mathbf{v}$ is in the span of S.

6. Show that the vectors $(1,1,0)$, $(1,0,1)$, and $(0,1,1)$ generate F^3.

Here, F is the scalar field. We need to show that every vector (a,b,c) in F^3 is a linear combination of the three vectors. There are a couple approaches you could take.

One way is to show that the three standard vectors $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ are linear combinations. Since all vectors in F^3 can be generated by them, if we can generate them, we'll be able to generate all vectors. (This depends on the theorem: *a linear combination of linear combinations is a linear combination.*)

So, how can you get $(1,0,0)$? You can find it by the techniques of the earlier exercises or just look for it. You'll find $(1,0,0) = \frac{1}{2}((1,1,0)+(1,0,1)-(0,1,1))$. The other two standard vectors are similarly two of the given vectors minus the third.

An alternate way is to show that an arbitrary vector (a,b,c) in F^3 is a linear combination of the three vectors. In other words, solve the vector equation

$$x(1,1,0)+y(1,0,1)+z(0,0,1)=(a,b,c)$$

for x, y, and z in terms of a, b, and c.

9. Show that the four matrices $\begin{bmatrix}1&0\\0&0\end{bmatrix}$, $\begin{bmatrix}0&1\\0&0\end{bmatrix}$, $\begin{bmatrix}0&0\\1&0\end{bmatrix}$, and $\begin{bmatrix}0&0\\0&1\end{bmatrix}$ generate $M_{2\times 2}(F)$.

This is pretty easy. All you have to do is show that a generic matrix $\begin{bmatrix}a&b\\c&d\end{bmatrix}$ is a linear combination of those four matrices. But it is since it equals

$$a\begin{bmatrix}1&0\\0&0\end{bmatrix}+b\begin{bmatrix}0&1\\0&0\end{bmatrix}+c\begin{bmatrix}0&0\\1&0\end{bmatrix}+d\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

14. Show that if S_1 and S_2 are arbitrary subsets of a vector space V then the span of their union, $\operatorname{span}(S_1\cup S_2)$, equals the sum of their spans, $\operatorname{span}(S_1)+\operatorname{span}(S_2)$.

An element of $\operatorname{span}(S_1\cup S_2)$ is a linear combination of vectors in $S_1\cup S_2$, while an element of $\operatorname{span}(S_1)+\operatorname{span}(S_2)$ is the sum of two vectors, one being a lin combo of vectors in S_1, the other being a lin combo of vectors in S_2.

But a lin combo of vectors in $S_1\cup S_2$ is the sum of a lin combo of vectors in S_1 and a lin combo of vectors in S_2, and vice versa.

We can add details to the argument as follows. Let the vector $\mathbf{w}$ be the linear combination of vectors in $S_1\cup S_2$, and arrange the terms so that the vectors in S_1 come before the vectors in S_2.

$$\mathbf{w}=c_1\mathbf{v}_1+\cdots+c_k\mathbf{v}_k+c_{k+1}\mathbf{v}_{k+1}+\cdots+c_n\mathbf{v}_n$$

where $\mathbf{v}_1,\ldots,\mathbf{v}_k\in S_1$ and $\mathbf{v}_{k+1},\ldots,\mathbf{v}_n\in S_2$. Then $\mathbf{w}$ is the sum of the two vectors, $c_1\mathbf{v}_1+\cdots+c_k\mathbf{v}_k$ in the span of S_1, and $c_{k+1}\mathbf{v}_{k+1}+\cdots+c_n\mathbf{v}_n$ in the span of S_2. And, of course, the sum of two such vectors is a vector in the span of $S_1\cup S_2$.

The primary advantage of the more detailed proof is that it emphasizes that the terms in the sum need to be rearranged to express the sum $\mathbf{w}$ as the sum of two vectors, one from $\operatorname{span}(S_1)$ and the other from $\operatorname{span}(S_2)$.

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