The definition of cross products. The cross product \( \times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \) is an operation that takes two vectors \( \mathbf{u} \) and \( \mathbf{v} \) in space and determines another vector \( \mathbf{u} \times \mathbf{v} \) in space. (Cross products are sometimes called outer products, sometimes called vector products.) Although we’ll define \( \mathbf{u} \times \mathbf{v} \) algebraically, its geometric meaning is understandable. The vector \( \mathbf{u} \times \mathbf{v} \) will have a length equal to the area of the parallelogram whose sides are \( \mathbf{u} \) and \( \mathbf{v} \), and the direction of \( \mathbf{u} \times \mathbf{v} \) will be orthogonal to the plane of \( \mathbf{u} \) and \( \mathbf{v} \) in a direction determined by a right-hand rule (when the coordinate system is right-handed).

The easiest way to define cross products is to use the standard unit vectors \( \mathbf{i}, \mathbf{j}, \) and \( \mathbf{k} \) for \( \mathbb{R}^3 \). If

\[
\mathbf{u} = (u_1, u_2, u_3) = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k},
\]

and

\[
\mathbf{v} = (v_1, v_2, v_3) = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k},
\]

then \( \mathbf{u} \times \mathbf{v} \) is defined as

\[
\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}
\]

which is much easier to remember when you write it as a determinant

\[
\mathbf{u} \times \mathbf{v} = \begin{vmatrix}
  u_2 & u_3 \\
  v_2 & v_3
\end{vmatrix} \mathbf{i} - \begin{vmatrix}
  u_1 & u_3 \\
  v_1 & v_3
\end{vmatrix} \mathbf{j} + \begin{vmatrix}
  u_1 & u_2 \\
  v_1 & v_2
\end{vmatrix} \mathbf{k}
\]

Properties of cross products. There are a whole lot of properties that follow from this definition. First of all, it’s anticommutative

\[
\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}),
\]

so any vector cross itself is \( \mathbf{0} \)

\[
\mathbf{v} \times \mathbf{v} = \mathbf{0}.
\]

It’s bilinear, that is, linear in each argument, so it distributes over addition and subtraction, \( \mathbf{0} \) acts as zero should, and you can pass scalars in and out of arguments

\[
\mathbf{u} \times (\mathbf{v} \pm \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \pm (\mathbf{u} \times \mathbf{w})
\]

\[
(\mathbf{u} \pm \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) \pm (\mathbf{v} \times \mathbf{w})
\]
\[ 0 \times v = 0 = v \times 0 \]

\[ c(u \times v) = (cu) \times v = u \times (cv) \]

A couple more properties you can check from the definition, or from the properties already found are that \( \langle u \times v \mid u \rangle = 0 \) and \( \langle u \times v \mid v \rangle = 0 \). Those imply that the vector \( u \times v \) is orthogonal to both vectors \( u \) and \( v \), and so it is orthogonal to the plane of \( u \) and \( v \).

**Standard unit vectors and cross products.** Interesting things happen when we look specifically at the cross products of standard unit vectors. Of course

\[ i \times i = j \times j = k \times k = 0, \]

since any vector cross itself is 0. But

\[ i \times j = k, \quad j \times k = i, \quad k \times i = j, \]

and

\[ j \times i = -k, \quad k \times j = -i, \quad i \times k = -j, \]

all of which follows directly from the definition.

**Length of the cross product, areas of triangles and parallelograms.** A direct computation (which we’ll omit) shows that

\[ \|u \times v\| = \|u\| \|v\| \sin \theta \]

where \( \theta \) is the angle between the vectors \( u \) and \( v \).

Consider a triangle in 3-space where two of the sides are \( u \) and \( v \).

![Triangle Diagram](image)

Taking \( u \) to be the base of the triangle, then the height of the triangle is \( \|v\| \sin \theta \), where \( \theta \) is the angle between \( u \) and \( v \). Therefore, the area of this triangle is

\[ \text{Area} = \frac{1}{2} \|u\| \|v\| \sin \theta = \frac{1}{2} \|u \times v\|. \]

(In general, the area of a any triangle is half the product of two adjacent sides and the sine of the angle between them.)
Area of a parallelogram in $\mathbb{R}^3$. Now consider a parallelogram in 3-space where two of the sides are $\mathbf{u}$ and $\mathbf{v}$.

Of course, if the triangle is doubled to a parallelogram, then the area of the parallelogram is $\|\mathbf{u} \times \mathbf{v}\|$.

Thus, the norm of a cross product is the area of the parallelogram bounded by the vectors.

We now have a geometric characterization of the cross product. The cross product $\mathbf{u} \times \mathbf{v}$ is the vector orthogonal to the plane of $\mathbf{u}$ and $\mathbf{v}$ pointing away from it in a the direction determined by a right-hand rule, and its length equals the area of the parallelogram whose sides are $\mathbf{u}$ and $\mathbf{v}$.

Note that $\mathbf{u} \times \mathbf{v}$ is $\mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ lie in a line, that is, they point in the same direction or the directly opposite directions.

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