

Determinants, part II  
 Math 130 Linear Algebra  
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So far we've only defined determinants of  $2 \times 2$  and  $3 \times 3$  matrices. The  $2 \times 2$  determinants had 2 terms, while the  $3 \times 3$  determinants had 6 terms.

There are many ways that general  $n \times n$  determinants can be defined. We'll first define a determinant function in terms of characterizing properties that we want it to have. Then we'll use the construction of a determinant following the method given in the text.

**Definition 1.** A *determinant function* assigns to each square matrix  $A$  a scalar associated to the matrix, denoted  $\det(A)$  or  $|A|$  such that

1. The determinant of an  $n \times n$  identity matrix  $I$  is 1.  $|I| = 1$ .
2. If the matrix  $B$  is identical to the matrix  $A$  except the entries in one of the rows of  $B$  are each equal to the corresponding entries of  $A$  multiplied by the same scalar  $c$ , then  $|B| = c|A|$ .
3. If the matrices  $A$ ,  $B$ , and  $C$  are identical except for the entries in one row, and for that row an entry in  $A$  is found by adding the corresponding entries in  $B$  and  $C$ , then  $|A| = |B| + |C|$ .
4. If the matrix  $B$  is the result of exchanging two rows of  $A$ , then the determinant of  $B$  is the negation of the determinant of  $A$ .

The second and third conditions together say that determinants are linear in each row. Usually that phrased by saying determinants are multilinear in their rows.

The last condition says it's an "alternating" function.

These conditions are enough to characterize the determinant, but they don't show such a determinant function exists and is unique. We'll show both existence and uniqueness, but start with uniqueness. First, we'll note a couple of properties that determinant functions have that follow from the definition.

**Theorem 2.** A determinant function has the following two properties.

- (a). The determinant of any matrix with an entire row of 0's is 0.
- (b). The determinant of any matrix with two identical rows is 0.

*Proof.* Property (a) follows from the second statement in the definition. If  $A$  has a whole row of 0's, then using that row and  $c = 0$  in the second statement of the definition, then  $B = A$ , so  $|A| = 0|A|$ . Therefore,  $|A| = 0$ .

Property (b) follows from the fourth statement in the definition. If you exchange the two identical rows, the result is the original matrix, but its determinant is negated. The only scalar which is its own negation is 0. Therefore, the determinant of the matrix is 0. Q.E.D.

Now we can show the uniqueness of determinant functions.

**Theorem 3.** There is at most one determinant function.

*Proof.* The four properties that determinants have are enough to find the value of the determinant of a matrix.

Suppose a matrix  $A$  has more than one nonzero entry in a row. Then using the sum property of multilinearity, we know

$$|A| = |A_1| + |A_2| + \cdots + |A_n|$$

where  $A_j$  is the matrix that looks just like  $A$  except in that row, all the entries are 0 except the  $j^{\text{th}}$  one which is the  $j^{\text{th}}$  entry of that row in  $A$ .

That means we can reduce the question of evaluating determinants of general matrices to evaluating determinants of matrices that have at most one nonzero entry in each row.

By property (a) in the preceding theorem, if the matrix has a row of all 0's, its determinant is 0. Thus, we only need to consider matrices that have exactly one nonzero entry in each row.

Using the scalar property of multilinearity, we can further assume that the nonzero entry in that row is 1.

Now we're down to evaluating determinants that only have entries of 0's and 1's with exactly one 1 in each row.

If two of those rows have the 1's in the same column, then by property (b) of the preceding theorem, that matrix has determinant 0.

Now the only matrices left to consider are those that only have entries of 0's and 1's with exactly one 1 in each row and column. These are called *permutation matrices*.

Using alternation, the fourth condition, the rows can be interchanged until the 1's only lie on the main diagonal, and that negates the value of the matrix for interchange we make.

Finally, we're left with the identity matrix, but by the first condition in the definition, its determinant is 0.

Thus, the value of the determinant of every matrix is determined by the definition. There can be only one determinant function. Q.E.D.

Although that argument shows that there's at most one function with those four properties, it doesn't show that there is such a function. In other words, we haven't shown the properties are consistent. We need some way to construct a function with those properties, and we'll do that with a "cofactor construction".

In the process of proving that theorem, we found out what the determinant of a permutation matrix had to be. Let's state that as a corollary since it's an important result in its own right.

**Corollary 4.** A *permutation matrix* is a square matrix that only has 0's and 1's as its entries with exactly one 1 in each row and column.

The determinant of a permutation matrix will have to be either 1 or  $-1$  depending on whether it takes an even number or an odd number of row interchanges to convert it to the identity matrix.

**The cofactor construction.** Although there are many ways to construct the determinant function, we'll use the cofactor construction. Another common construction uses permutations.

**Example 5.** Probably the best way to understand cofactor expansion is to use it in an example. In this example, we'll do cofactor expansion across the first row of a matrix. Let's start with a typical  $3 \times 3$  matrix

$$A = \begin{bmatrix} 5 & 2 & -1 \\ 4 & -2 & 3 \\ 6 & 1 & 4 \end{bmatrix}$$

For each of the three entries across the first row, we'll cross out the row and column of that entry to get a  $2 \times 2$  matrix called a *cofactor matrix*. Then we'll multiply each entry in the first row by the determinant of its cofactor matrix, and then add or subtract the resulting products to get the determinant.

$$|A| = 5 \cdot \begin{vmatrix} -2 & 3 \\ 1 & 4 \end{vmatrix} - 2 \cdot \begin{vmatrix} 4 & 3 \\ 6 & 4 \end{vmatrix} + (-1) \cdot \begin{vmatrix} 4 & -2 \\ 6 & 1 \end{vmatrix}$$

Thus, we've reduced the problem of finding one  $3 \times 3$  determinant to finding three  $2 \times 2$  determinants. The rest of the computation goes like this:

$$|A| = 5 \cdot (-11) - 2 \cdot (-2) + (-1) \cdot 16 = -55 + 4 - 16 = -67$$

Now, notice how we alternately added and subtracted the three terms to get the product. When you expand across the first row, the signs you use are  $+ - +$ . You could instead expand across the second row, you'll use the signs  $- + -$ . The sign you use for the  $i_j^{\text{th}}$  entry is  $(-1)^{i+j}$ . This corresponds to this checkerboard pattern for the entries of the matrix

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

which starts with  $+$  in the upper left corner.

We'll construct a determinant function using expansion across the first row of the matrix. We'll call this function we're constructing a determinant and use the standard notation for it, but we have to show that it has the four properties required for a determinant function.

This cofactor construction is an recursive definition (also called an inductive definition) since the determinant of an  $n \times n$  matrix is defined in terms of determinants of  $(n-1) \times (n-1)$  matrices. Like all recursive definitions, we need a base case, and that will be when  $n = 1$ . We'll define the determinant of a  $1 \times 1$  matrix to be its entry.

We need to define cofactor matrices before we can get to the construction.

**Definition 6.** If  $A$  is an  $n \times n$  matrix,  $i$  is a row number, and  $j$  is a column number, then the  $i_j^{\text{th}}$  cofactor matrix, denoted  $\tilde{A}_{ij}$ , is the matrix that results from deleting the  $i^{\text{th}}$  row and the  $j^{\text{th}}$  column from  $A$ . The cofactor matrix  $\tilde{A}_{ij}$  is an  $(n-1) \times (n-1)$  matrix.

Our recursive construction for determinants “expands” the matrix along the first row. We define  $|A|$  as an alternating sum of the determinants of its cofactor matrices

$$|A| = \sum_{j=1}^n (-1)^{1+j} A_{1j} |\tilde{A}_{1j}| = A_{11} |\tilde{A}_{11}| - A_{12} |\tilde{A}_{12}| + A_{13} |\tilde{A}_{13}| - \cdots \pm A_{1n} |\tilde{A}_{1n}|$$

This definition gives the same values for  $2 \times 2$  and  $3 \times 3$  determinants we saw before.

The following theorem is formally proven in steps in the text. The proof here is complete except one step which will be illustrated in the case  $n = 3$ .

**Theorem 7.** The four characterizing properties of determinants listed above are satisfied by the cofactor definition of determinants.

*Proof.* We’ll check the four properties of a determinant function one at a time.

1. The determinant of an  $n \times n$  identity matrix  $I$  is 1.  $|I| = 1$ .

It’s easy to check that with this construction, the determinant of the identity matrix is 1.

2. If the matrix  $B$  is identical to the matrix  $A$  except the entries in one of the rows of  $B$  are each equal to the corresponding entries of  $A$  multiplied by the same scalar  $c$ , then  $|B| = c|A|$ .

We’ll have to consider two cases, one for the first row, the other for any other row.

For the first row, the entries of the first row of  $B$  are  $c$  times the entries of the first row of  $A$ , that is  $B_{1j} = cA_{1j}$ , while the cofactor matrices  $\tilde{B}_{1j}$  are the same as the cofactors matrices  $\tilde{A}_{1j}$ . Therefore,

$$|B| = \sum_{j=1}^n (-1)^{1+j} B_{1j} |\tilde{B}_{1j}| = \sum_{j=1}^n (-1)^{1+j} cA_{1j} |\tilde{A}_{1j}| = c|A|.$$

Suppose now that the entries of the  $r^{\text{th}}$  row are multiplied by  $c$ , where  $r$  is some row number greater than 1. This time the first row of  $B$  is the same as the first row of  $A$ , so  $B_{1j} = A_{1j}$ , but the matrices  $\tilde{B}_{1j}$  each have a row, the  $r - 1^{\text{st}}$  row, which is  $c$  times the same row in  $\tilde{A}_{1j}$ . By induction we may assume their determinants are each multiplied by  $c$ , that is,  $|\tilde{B}_{1j}| = c|\tilde{A}_{1j}|$ . Thus

$$|B| = \sum_{j=1}^n (-1)^{1+j} B_{1j} |\tilde{B}_{1j}| = \sum_{j=1}^n (-1)^{1+j} A_{1j} c |\tilde{A}_{1j}| = c|A|.$$

3. If the matrices  $A$ ,  $B$ , and  $C$  are identical except for the entries in one row, and for that row an entry in  $A$  is found by adding the corresponding entries in  $B$  and  $C$ , then  $|A| = |B| + |C|$ .

Again, we’ll have to consider two cases, one for the first row, the other for any other row.

For the first row, the entries in the first row of  $A$  are the sums of the corresponding entries in the first row of  $B$  and  $C$ , that is,  $A_{1j} = B_{1j} + C_{1j}$ , but all the rest of the rows are the same in  $A$ ,  $B$ , and  $C$ . Then the cofactor matrices for  $A$ ,  $B$ , and  $C$  are the same:  $\tilde{A}_{1j} = \tilde{B}_{1j} = \tilde{C}_{1j}$ . Therefore,

$$\begin{aligned}
|A| &= \sum_{j=1}^n (-1)^{1+j} A_{1j} |\tilde{A}_{1j}| \\
&= \sum_{j=1}^n (-1)^{1+j} (B_{1j} + C_{1j}) |\tilde{A}_{1j}| \\
&= \sum_{j=1}^n (-1)^{1+j} B_{1j} |\tilde{A}_{1j}| + \sum_{j=1}^n (-1)^{1+j} C_{1j} |\tilde{A}_{1j}| \\
&= \sum_{j=1}^n (-1)^{1+j} B_{1j} |\tilde{B}_{1j}| + \sum_{j=1}^n (-1)^{1+j} C_{1j} |\tilde{B}_{1j}| = |B| + |C|
\end{aligned}$$

Suppose now for  $r > 1$  that the entries of the  $r^{\text{th}}$  row of  $A$  are the sums of the corresponding entries in the  $r^{\text{th}}$  row of  $B$  and  $C$ , that is,  $A_{rj} = B_{rj} + C_{rj}$ , but all the rest of the rows are the same in  $A$ ,  $B$ , and  $C$ . Then the entries in the first row are the same,  $A_{1j} = B_{1j} = C_{1j}$ , and the cofactor matrix  $\tilde{A}_{1j}$  will be the same as the cofactor matrices  $\tilde{B}_{1j}$  and  $\tilde{C}_{1j}$  except that the  $r - 1^{\text{st}}$  row of  $\tilde{A}_{1j}$  will be sum of the  $r - 1^{\text{st}}$  rows of  $\tilde{B}_{1j}$  and  $\tilde{C}_{1j}$ . By induction, we may assume  $|\tilde{A}_{1j}| = |\tilde{B}_{1j}| + |\tilde{C}_{1j}|$ . Therefore,

$$\begin{aligned}
|A| &= \sum_{j=1}^n (-1)^{1+j} A_{1j} |\tilde{A}_{1j}| \\
&= \sum_{j=1}^n (-1)^{1+j} A_{1j} (|\tilde{B}_{1j}| + |\tilde{C}_{1j}|) \\
&= \sum_{j=1}^n (-1)^{1+j} A_{1j} |\tilde{B}_{1j}| + \sum_{j=1}^n (-1)^{1+j} A_{1j} |\tilde{C}_{1j}| \\
&= \sum_{j=1}^n (-1)^{1+j} B_{1j} |\tilde{B}_{1j}| + \sum_{j=1}^n (-1)^{1+j} C_{1j} |\tilde{C}_{1j}| = |B| + |C|
\end{aligned}$$

4. If the matrix  $B$  is the result of exchanging two rows of  $A$ , then the determinant of  $B$  is the negation of the determinant of  $A$ .

One more time, we'll have to consider two cases. In one case, neither of the two rows being exchanged is the first row. In that case, by induction, since the determinants of all the matrices  $\tilde{A}_{1j}$  are negated, then the determinant of  $A$  will be negated.

That leaves the case when one of the two rows being exchange is the first. We can reduce this case to the subcase when the other row is the second, since you can exchange the first and the  $r^{\text{th}}$  rows by three exchanges: first exchange the second and the  $r^{\text{th}}$ , then the first and the second, then the second and the  $r^{\text{th}}$  again.

Now we're down to the subcase where we exchange the first and the second rows of  $A$ . We'll show that exchange negates the determinant. We'll expand  $A$  twice, first along the first row of  $A$ , then along the first row of each of  $A_{1j}$ , which, of course, comes from the second row of  $A$ . This gets difficult to write formally for general  $n$ , so instead, we'll take  $n = 3$  which may be large enough to see how it works for larger  $n$ .

$$\begin{aligned}
 |A| &= \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \\
 &= A_{11} \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} - A_{12} \begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} + A_{13} \begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix} \\
 &= A_{11}(A_{22}|A_{33}| - A_{23}|A_{32}|) - A_{12}(A_{21}|A_{33}| - A_{23}|A_{31}|) + A_{13}(A_{21}|A_{32}| - A_{22}|A_{31}|) \\
 &= (A_{11}A_{22} - A_{12}A_{21})|A_{33}| \\
 &\quad + (A_{12}A_{23} - A_{13}A_{22})|A_{31}| \\
 &\quad + (A_{13}A_{21} - A_{11}A_{23})|A_{32}|
 \end{aligned}$$

Now, to determine what  $|B|$  is, all we have to do is interchange all the 1's and 2's for any of the first subscripts in that last computation. (3's and any larger subscripts for general  $n$  won't change.) We'll find

$$\begin{aligned}
 |B| &= (A_{21}A_{12} - A_{22}A_{11})|A_{33}| \\
 &\quad + (A_{22}A_{13} - A_{23}A_{12})|A_{31}| \\
 &\quad + (A_{23}A_{11} - A_{21}A_{13})|A_{32}|
 \end{aligned}$$

But that's the negation of the last expression for the computation for  $|A|$ , so  $|B| = -|A|$ .

Thus, this cofactor construction satisfies all four properties of a determinant function. Q.E.D.

The real purpose of the preceding theorem is just to show the existence of a determinant function. We usually won't use the construction to actually compute determinants because there are much better ways. There are some special matrices where this construction gives a quick method of evaluation, namely diagonal, upper triangular, and lower triangular matrices.

**Theorem 8.** The determinant of a diagonal matrix or a triangular matrix is the product of its diagonal entries.

*Proof.* In each of these three kinds of matrices, if you use cofactor expansion along the first row, the only term in the expansion that's not zero is the first, and that term is the product of the first entry and its cofactor matrix, which by induction will have a determinant which is the product of its diagonal entries. Q.E.D.

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