Determinants, part III
Math 130 Linear Algebra

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We characterized what determinant functions are based on four properties, and we saw one construction for them. Here's a quick summary of their properties. The first four characterize them; the others we proved.

A *determinant function* assigns to each square matrix A a scalar associated to the matrix, denoted $\det(A)$ or $|A|$ such that

1. The determinant of an $n \times n$ identity matrix I is 1. $|I| = 1$.
2. If the matrix B is identical to the matrix A except the entries in one of the rows of B are each equal to the corresponding entries of A multiplied by the same scalar c, then $|B| = c\,|A|$.
3. If the matrices A, B, and C are identical except for the entries in one row, and for that row an entry in A is found by adding the corresponding entries in B and C, then $|A| = |B| + |C|$.
4. If the matrix B is the result of exchanging two rows of A, then the determinant of B is the negation of the determinant of A.
5. The determinant of any matrix with an entire row of 0's is 0.
6. The determinant of any matrix with two identical rows is 0.
7. There is one and only one determinant function.
8. The determinant of a permutation matrix is either 1 or -1 depending on whether it takes an even number or an odd number of row interchanges to convert it to the identity matrix.

Other properties of determinants. There are several other important properties of determinants.

For instance, determinants can be evaluated by cofactor expansion along any row, not just the first row as we used to construct the determinant. We won't take the time to prove that. The idea of the proof is straightforward—exchange the given row with the first row and apply cofactor expansion along the first row. The only difficultly with the proof is keeping track of the sign of the determinant.

As mentioned before, we won't use cofactor expansion much since it's not a practical way to evaluate determinants. Here are a couple of more useful properties.

Theorem 1. If one row of a square matrix is a multiple of another row, then its determinant is 0.

Proof. We saw that if two rows are the same, then a square matrix has 0 determinant. By the second property of determinants if we multiply one of those rows by a scalar, the matrix's determinant, which is 0, is multiplied by that scalar, so that determinant is also 0. Q.E.D.

Theorem 2. The determinant of a matrix is not changed when a multiple of one row is added to another.

Proof. Let A be the given matrix, and let B be the matrix that results if you add c times row k to row l, $k \neq l$. Let C be the matrix that looks just like A except the lth row of C is c times the kth row. Since one row of C is a multiple of another row of C, its determinant is 0. By multilinearity of the determinant (property 3), $|A| = |B| + |C|$. Since $|C| = 0$, therefore $|A| = |B|$. Q.E.D.

An efficient algorithm for evaluating a matrix. The three row operations are (1) exchange rows, and that will negate the determinant, (2) multiply or divide a row by a nonzero constant, and that scales the determinant by that constant, and (3) add a multiple of one row to another, and that doesn't change the determinant at all.

After you've row reduced the matrix, you'll have a triangular matrix, and its determinant will be the product of its diagonal entries.

We'll evaluate a couple of matrices by this method in class.

Determinants, rank, and invertibility. There's a close connection between these for a square matrix. We've seen that an $n \times n$ matrix A has an inverse if and only if $\operatorname{rank}(A) = n$. We can add another equivalent condition to that, namely, $|A| \neq 0$.

Theorem 3. The determinant of an $n \times n$ matrix is nonzero if and only if its rank is n, that is to say, if and only if it's invertible.

Proof. The determinant of the matrix will be 0 if and only if when it's row reduced the resulting matrix has a row of 0s, and that happens when its rank is less than n. Q.E.D.

Determinant of products and inverses. These are rather important properties of determinants. The first says $|AB| = |A|\,|B|$, and the second says $|A^{-1}| = |A|^{-1}$ when A is an invertible matrix.

Theorem 4. The determinant of the product of two square matrices is the product of their determinants, that is, $|AB| = |A|\,|B|$.

Proof. We'll prove this in two cases, first when A has rank less than n, then when A has full rank. For the full rank case we'll reduce the proof to the case where A is an elementary case since it's easy to show the result in that case.

Case 1. Assume that the rank of A is less than n. Then by the previous theorem, $|A| = 0$. Since the rank of the product of two matrices is less than or equal to the rank of each, therefore the rank of AB is less than n, and, hence $|AB| = 0$. Thus, no matter what B is, $|AB| = |A|\,|B|$.

Case 2. For this case assume the rank of A is n. Then A can be expressed as a product of elementary matrices $A = E_1 E_2 \cdots E_k$. If we knew for each elementary matrix E that $|EB| = |E|\,|B|$, then it would follow that

$$\begin{aligned} |AB| &= |E_1 E_2 \cdots E_k B| \\ &= |E_1|\,|E_2| \cdots |E_k|\,|B| \\ &= |A|\,|B| \end{aligned}$$

Thus, we can reduce case 2 to the special case where A is an elementary matrix.

Elementary subcases. We'll show that for each elementary matrix E that $|EB| = |E|\,|B|$. There are three kinds of elementary matrices, and we'll check each one.

Elementary subcase 1. Suppose that E is the result of interchanging two rows of the identity matrix I. Then $|E| = -1$, and EB is the same as B except those two rows are interchanged, so $|EB| = -|EB|$. Thus, $|EB| = |E|\,|B|$.

Elementary subcase 2. Suppose that E is the result of multiplying a row of I by a scalar c. Then $|E| = c$, and EB is the same as B except that row is multiplied by c, so $|EB| = c\,|B|$. Thus, $|EB| = |E|\,|B|$.

Elementary subcase 3. Suppose that E is the result of adding a multiple of one row of I to another. Then $|E| = 1$, and EB is the same as B except that that same multiple of one row of B is added the same other of B, so $|EB| = |B|$. Again, $|EB| = |E|\,|B|$. Q.E.D.

Corollary 5. For an invertible matrix, the determinant of its inverse is the reciprocal of its determinant, that is, $|A^{-1}| = |A|^{-1}$.

Proof. According to the theorem, $|A^{-1}A| = |A|\,|A^{-1}|$, but $|A^{-1}A| = |I| = 1$, so $|A|\,|A^{-1}| = 1$ from which it follows that $|A^{-1}| = |A|^{-1}$. Q.E.D.

More generally, $|A^p| = |A|^p$ for general p even when p is negative so long as A is an invertible matrix.

Determinants and transposes. So far, everything we've said about determinants of matrices was related to the rows of the matrix, so it's somewhat surprising that a matrix and its transpose have the same determinant. We'll prove that, and from that theorem we'll automatically get corresponding statements for columns of matrices that we have for rows of matrices.

Theorem 6. The determinant of the transpose of a square matrix is equal to the determinant of the matrix, that is, $|A^t| = |A|$.

Proof. We'll prove this like the last theorem. First in the case where the rank of A is less than n, then the case where the rank of A is n, and for the second case we'll write A as a product of elementary matrices.

Case 1. Assume that the rank of A is less than n. Then its determinant is 0. But the rank of a matrix is the same as the rank of its transpose, so A^t has rank less than n and its determinant is also 0.

Case 2. For this case assume the rank of A is n. Express A as a product of elementary matrices, $A = E_1 E_2 \cdots E_k$. If we knew for each elementary matrix E that $|E^t| = |E|$, then it would follow that

$$\begin{aligned} |A| &= |E_1 E_2 \cdots E_k| \\ &= |E_1|\,|E_2| \cdots |E_k| \\ &= |E_1^t|\,|E_2^t| \cdots |E_k^t| \\ &= |E_1^t E_2^t \cdots E_k^t| = |A^t| \end{aligned}$$

(Note how we used the property that the transpose of a product equals the product of the traposes.)

Thus, we can reduce case 2 to the special case where A is an elementary matrix. The details that $|E^t| = |E|$ for each of the three kinds of elementary matrices are omitted here since that's easy to verify.

Q.E.D.

The following properties of determinants that relate to the columns of a matrix follow from this theorem and the corresponding properties for rows of a matrix.

1. If the matrix B is identical to the matrix A except the entries in one of the columns of B are each equal to the corresponding entries of A multiplied by the same scalar c, then $|B| = c\,|A|$.

2. If the matrices A, B, and C are identical except for the entries in one column, and for that column an entry in A is found by adding the corresponding entries in B and C, then $|A| = |B| + |C|$.

3. If the matrix B is the result of exchanging two columns of A, then the determinant of B is the negation of the determinant of A.

4. The determinant of any matrix with an entire column of 0's is 0.

5. The determinant of any matrix with two identical columns is 0.

6. The determinant of a permutation matrix is either 1 or -1 depending on whether it takes an even number or an odd number of column interchanges to convert it to the identity matrix.

7. The determinant of a matrix can be evaluated by cofactor expansion along any column.

Cramer's rule. This is a method based on determinants to find the solution to a system of n equations in n unknowns when there is exactly one solution. The solution is has the determinant in the denominator, and the only time the determinant is not zero is when there's a unique solution.

Cramer's rule is one of the oldest applications of determinants. It's not an efficient method to solve a system since row reduction is faster, but it's an interesting use of determinants.

Here's an example to show how to apply Cramer's rule. Let's suppose we have the follow-

ing system of three equations in three unknowns.

$$\begin{array}{rcrcrcr} x & + & y & + & 3z & = & 6 \\ 2x & + & 3y & - & 4z & = & -2 \\ 3x & - & 2y & + & 5z & = & 7 \end{array}$$

First, compute the determinant Δ of the 3×3 coefficient matrix.

$$\Delta = \begin{vmatrix} 1 & 1 & 3 \\ 2 & 3 & -4 \\ 3 & -2 & 5 \end{vmatrix} = -54$$

Next, replace the first column by the constant vector, and compute that determinant.

$$\Delta_x = \begin{vmatrix} 6 & 1 & 3 \\ -2 & 3 & -4 \\ 7 & -2 & 5 \end{vmatrix} = -27$$

Then in the unique solution, $x = \Delta_x/\Delta = \frac{1}{2}$. Next, replace the second column by the constant vector, and compute that determinant.

$$\Delta_y = \begin{vmatrix} 1 & 6 & 3 \\ 2 & -2 & -4 \\ 3 & 7 & 5 \end{vmatrix} = -54$$

So $y = \Delta_y/\Delta = 1$. Likewise, replace the third column by the constant vector.

$$\Delta_z = \begin{vmatrix} 1 & 1 & 6 \\ 2 & 3 & -2 \\ 3 & -2 & 7 \end{vmatrix} = -81$$

which gives $z = \frac{3}{2}$. Thus, the unique solution is $(x, y, z) = (\frac{1}{2}, 1, \frac{3}{2})$.

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