CLARK UNIVERSITY

Diagonalizable operators
Math 130 Linear Algebra

D Joyce, Fall 2013

Some linear operators $T : V \to V$ have the nice property that there is some basis for V so that the matrix representing T is a diagonal matrix. We'll call those operators *diagonalizable operators*. We'll call a square matrix A a *diagonalizable matrix* if it is conjugate to a diagonal matrix, that is, there exists an invertible matrix P so that $P^{-1}AP$ is a diagonal matrix. That's the same as saying that under a change of basis, A becomes a diagonal matrix.

Reflections are examples of diagonalizable operators as are rotations if $\mathbf{C}$ is your field of scalars.

Not all linear operators are diagonalizable. The simplest one is $\mathbf{R}^2 \to \mathbf{R}^2$, $(x, y) \to (y, 0)$ whose matrix is $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. No conjugate of it is diagonal. It's an example of a *nilpotent* matrix, since some power of it, namely A^2, is the 0-matrix. In general, nilpotent matrices aren't diagonalizable. There are many other matrices that aren't diagonalizable as well.

Theorem 1. A linear operator on an n-dimensional vector space is diagonalizable if and only if it has a basis of n eigenvectors, in which case the diagonal entries are the eigenvalues for those eigenvectors.

Proof. If it's diagonalizable, then there's a basis for which the matrix representing it is diagonal. The transformation therefore acts on the i^{th} basis vector by multiplying it by the i^{th} diagonal entry, so it's an eigenvector. Thus, all the vectors in that basis are eigenvectors for their associated diagonal entries.

Conversely, if you have a basis of n eigenvectors, then the matrix representing the transformation is diagonal since each eigenvector is multiplied by its associated eigenvalue. Q.E.D.

We'll see soon that if a linear operator on an n-dimensional space has n distinct eigenvalues, then it's diagonalizable. But first, a preliminary theorem.

Theorem 2. Eigenvectors that are associated to distinct eigenvalues are independent. That is, if $\lambda_1, \lambda_2, \ldots, \lambda_k$ are different eigenvalues of an operator T, and an eigenvector $\mathbf{v}_i$ is associated to each eigenvalue λ_i, then the set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is linearly independent.

Proof. Assume by induction that the first $k-1$ of the eigenvectors are independent. We'll show all k of them are. Suppose some linear combination of all k of them equals $\mathbf{0}$:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

Take $T - \lambda_k I$ of both sides of that equation. The left side simplifies

$$\begin{aligned} & (T-\lambda_k I)(c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k) \\ = \; & c_1T(\mathbf{v}_1) - \lambda_k c_1\mathbf{v}_1 + \cdots + c_kT(\mathbf{v}_k) - \lambda_k c_k\mathbf{v}_k \\ = \; & c_1(\lambda_1 - \lambda_k)\mathbf{v}_1 + \cdots + c_k(\lambda_k - \lambda_k)\mathbf{v}_k \\ = \; & c_1(\lambda_1 - \lambda_k)\mathbf{v}_1 + \cdots + c_{k-1}(\lambda_{k-1} - \lambda_k)\mathbf{v}_{k-1} \end{aligned}$$

and, of course, the right side is $\mathbf{0}$. That gives us a linear combination of the first $k-1$ vectors which equals $\mathbf{0}$, so all their coefficients are 0:

$$c_1(\lambda_1 - \lambda_k) = \cdots = c_{k-1}(\lambda_{k-1} - \lambda_k) = 0$$

Since λ_k does not equal any of the other λ_i's, therefore all the c_i's are 0:

$$c_1 = \cdots = c_{k-1} = 0$$

The original equation now says $c_k\mathbf{v}_k = \mathbf{0}$, and since the eigenvector $\mathbf{v}_k$ is not 0, therefore $c_k = 0$. Thus all k eigenvectors are linearly independent. Q.E.D.

Corollary 3. If a linear operator on an n-dimensional vector space has n distinct eigenvalues, then it's diagonalizable.

Proof. Take an eigenvector for each eigenvalue. By the preceding theorem, they're independent, and since there are n of them, they form a basis of the n-dimensional vector space. The matrix representing the transformation with respect to this basis is diagonal and has the eigenvalues displayed down the diagonal. Q.E.D.

Math 130 Home Page at
`http://math.clarku.edu/~djoyce/ma130/`