Dimension
Math 130 Linear Algebra

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We'd like to be able to say that the dimension of a vector space is the number of vectors in its basis, but we can't yet since we haven't yet proved that any two bases of a vector space have the same number of vectors. We'll do that, at least in the case that a vector space has a finite basis. If a vector space doesn't have a finite basis, it will have an infinite dimension. We've got enough to do just to with the finite dimensional ones.

The argument that shows two bases have the same size works by replacing the vectors of one with the vectors of the other. We'll state that separately and call it the Replacement Theorem. The desired theorem, that two bases have the same number of vectors, follows from that as a corollary.

Summary of points discussed so far. Recall that a subset S of a vector space V is linearly independent if $\mathbf{0}$ is not a nontrivial linear combination of the vectors in S. Equivalently, no vector in S is a linear combination of the rest of the vectors in S.

Also, recall that S is a *basis* of V if (1) it spans V, and (2) it is linearly independent. Equivalently, each vector in V may be uniquely represented as a linear combination of vectors from S.

We also saw that every finite spanning subset of a vector space has a subset which is a basis of that vector space.

The Replacement Theorem. The replacement theorem will allow us to replace vectors in a basis with other vectors to get a new basis. Its proof requires attention to detail.

Theorem 1 (Replacement)**.** If a basis of a vector space has n vectors, then no linearly independent set of vectors has more than n vectors.

Proof. Let $\beta_0 = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ be a basis of a vector space V, and let $T = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ be a set of linearly independent vectors in V. We need to show that $r \leq n$.

The idea is to replace the vectors in β_0 by vectors in T, one at a time, to get new bases, each with n vectors. Eventually, all the vectors in T will be part of a basis having n vectors, and that will show $r \leq n$.

Now, to get the replacement going right, add the vectors of T to β_0 one at a time. Suppose that we've done that for a while and we've replaced i of the vectors in β_0 with i of the vectors in V. Then we'll have a basis β_i of n vectors

$$\beta_i = \{\mathbf{w}_1, \ldots, \mathbf{w}_i, \mathbf{b}_{i+1}, \ldots, \mathbf{b}_n\}$$

The next step is to replace another of the $\mathbf{b}$'s with $\mathbf{w}_{i+1}$.

Since β_i is a basis, therefore $\mathbf{w}_{i+1}$ is a linear combination of vectors in β_i. It can't depend only on the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_i$ because the $\mathbf{w}$'s are independent, so that linear combination depends on one of the $\mathbf{b}$'s. Reindex the $\mathbf{b}$'s so a vector $\mathbf{w}_{i+1}$ that depends on is $\mathbf{b}_{i+1}$. We'll replace $\mathbf{v}_{i+1}$ with $\mathbf{w}_{i+1}$ to get

$$\beta_{i+1} = \{\mathbf{w}_1, \ldots, \mathbf{w}_{i+1}, \mathbf{b}_{i+2}, \ldots, \mathbf{b}_n\}$$

We have to show that β_{i+1} is a basis for V.

Claim. β_{i+1} spans V. We know every vector in V is a combo of vectors in β_i, and every vector in β_i is a vector in β_{i+1} except the vector $\mathbf{b}_{i+1}$ which is a combo of vectors in β_{i+1}, therefore every vector in V is a combo of vectors in β_{i+1}. Thus β_{i+1} spans V.

Claim. β_{i+1} is independent. Suppose you have $\mathbf{0}$ as a nontrivial linear combination of vectors of β_{i+1}. Since $\mathbf{w}_{i+1}$ is a linear combination of vectors in β_i, that would give a nontrivial linear combination of vectors in β_i. But β_i is independent, so that can't happen. Thus β_{i+1} is independent.

Therefore β_{i+1} is a basis for V.

Continue replacing vectors until all the vectors in T have replaced vectors of the original β_0. We end up with a basis of n vectors that includes all r vectors from T. Therefore $r \leq n$. Q.E.D.

Dimension of a vector space. The following corollary follows directly from the last theorem.

Corollary 2. Any two bases of a vector space have the same number of vectors (assuming at least one is a finite basis).

Proof. Since each basis is an independent set of vectors it has no more vectors than the other basis. Hence, the two bases have the same number of vectors. Q.E.D.

Definition 3. If a vector space as a finite basis, then the *dimension* of a vector space is the number of vectors in any of its bases.

For example, $\mathbf{R}^n$ has dimension n. Also, P_n, the space of polynomials in t of degree less than or equal to n, is a vector space of dimension $n+1$. Also, M_{mn}, the space of $m \times n$ matrices, has dimension mn.

More on bases, linear independence, and spanning sets. There are a few more observations we can make about bases and these related concepts.

Corollary 4. Any linearly independent set of vectors in a finite-dimensional vector space can be extended to a basis.

Proof. Take any basis and replace the vectors in that basis with the linearly independent vectors as done in the replacement theorem. You'll end up with a basis that includes the independent set. Q.E.D.

Corollary 5. If n vectors in a vector space of dimension n are independent, then they form a basis.

Proof. They can be extended to a basis of n vectors, but there are already n vectors, so it is itself that extension. That is to say, it's already a basis. Q.E.D.

Corollary 6. If n vectors in a vector space of dimension n span the space, then they form a basis.

Proof. We know that there's a subset S' of the n vectors S that form a basis for the space. But every basis of the space has n vectors. Therefore the subset S' has to be all of S. Q.E.D.

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