Eigenvalues, eigenvectors, and eigenspaces of linear operators
Math 130 Linear Algebra

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Eigenvalues and eigenvectors. We're looking at linear operators on a vector space V, that is, linear transformations $\mathbf{x} \mapsto T(\mathbf{x})$ from the vector space V to itself.

When V has finite dimension n with a specified basis β, then T is described by a square $n \times n$ matrix $A = [T]_\beta$.

We're particularly interested in the study the geometry of these transformations in a way that we can't when the transformation goes from one vector space to a different vector space, namely, we'll compare the original vector $\mathbf{x}$ to its image $T(\mathbf{x})$.

Some of these vectors will be sent to other vectors on the same line, that is, a vector $\mathbf{x}$ will be sent to a scalar multiple $\lambda\mathbf{x}$ of itself.

Definition 1. For a given linear operator $T : V \to V$, a nonzero vector $\mathbf{x}$ and a constant scalar λ are called an *eigenvector* and its *eigenvalue*, respectively, when $T(\mathbf{x}) = \lambda\mathbf{x}$. For a given eigenvalue λ, the set of all $\mathbf{x}$ such that $T(\mathbf{x}) = \lambda\mathbf{x}$ is called the λ-*eigenspace*. The set of all eigenvalues for a transformation is called its *spectrum*.

When the operator T is described by a matrix A, then we'll associate the eigenvectors, eigenvalues, eigenspaces, and spectrum to A as well. As A directly describes a linear operator on F^n, we'll take its eigenspaces to be subsets of F^n.

Theorem 2. Each λ-eigenspace is a subspace of V.

Proof. Suppose that $\mathbf{x}$ and $\mathbf{y}$ are λ-eigenvectors and c is a scalar. Then

$$T(\mathbf{x}+c\mathbf{y}) = T(\mathbf{x})+cT(\mathbf{y}) = \lambda\mathbf{x}+c\lambda\mathbf{y} = \lambda(\mathbf{x}+c\mathbf{y}).$$

Therefore $\mathbf{x} + c\mathbf{y}$ is also a λ-eigenvector. Thus, the set of λ-eigenvectors form a subspace of F^n. Q.E.D.

One reason these eigenvalues and eigenspaces are important is that you can determine many of the properties of the transformation from them, and that those properties are the most important properties of the transformation.

These are matrix invariants. Note that the eigenvalues, eigenvectors, and eigenspaces of a linear transformation were defined in terms of the transformation, not in terms of a matrix that describes the transformation relative to a particular basis. That means that they are invariants of square matrices under change of basis. Recall that if A and B represent the transformation with respect to two different bases, then A and B are conjugate matrices, that is, $B = P^{-1}AP$ where P is the transition matrix between the two bases. The eigenvalues are numbers, and they'll be the same for A and B. The corresponding eigenspaces will be isomorphic as subspaces of F^n under the linear operator of conjugation by P. Thus we have the following theorem.

Theorem 3. The eigenvalues of a square matrix A are the same as any conjugate matrix $B = P^{-1}AP$ of A. Furthermore, each λ-eigenspace for A is isomorphic to the λ-eigenspace for B. In particular, the dimensions of each λ-eigenspace are the same for A and B.

When 0 is an eigenvalue. It's a special situation when a transformation has 0 an an eigenvalue. That means $A\mathbf{x} = \mathbf{0}$ for some nontrivial vector $\mathbf{x}$. In other words, A is a singular matrix, that is, a matrix without an inverse. Thus,

Theorem 4. A square matrix is singular if and only if 0 is one of its eigenvalues. Put another way, a square matrix is invertible if and only if 0 is not one of its eigenvalues.

An example transformation that has 0 as an eigenvalue is a projection, like $(x, y, z) \mapsto (x, y, 0)$ that maps space to the xy-plane. For this projection, the 0-eigenspace is the z-axis.

In general, a 0-eigenspaces is the solution space of the homogeneous equation $A\mathbf{x} = \mathbf{0}$, what we've been calling the null space of A, and its dimension we've been calling the nullity of A.

When 1 is an eigenvalue. This is another important situation. It means the transformation has a subspace of fixed points. That's because vector $\mathbf{x}$ is in the 1-eigenspace if and only if $A\mathbf{x} = \mathbf{x}$.

An example transformation that has 1 as an eigenvalue is a reflection, like $(x, y, z) \mapsto (x, y, -z)$ that reflects space across the xy-plane. Its 1-eigenspace, that is, its subspace of fixed points, is the xy-plane. We'll look at reflections in $\mathbf{R}^2$ in detail in a moment.

Another transformation with 1 as an eigenvalue is the shear transformation $(x, y) \mapsto (x + y, y)$. Its 1-eigenspace is the x-axis.

Eigenvalues of reflections in $\mathbf{R}^2$. We've looked at reflections across some lines in the plane. There's a general form for a reflection across the line of slope $\tan\theta$, that is, across the line that makes an angle of θ with the x-axis. Namely, the matrix transformation $\mathbf{x} \mapsto A\mathbf{x}$, where

$$A = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix},$$

describes a such a reflection.

A reflection has fixed points, namely, the points on the line being reflected across. Therefore, 0 is an eigenvalue of a reflection, and the 0-eigenspace is that line.

Orthogonal to that line is a line passing through the origin and its points are reflected across the origin, that is to say, they're negated. Therefore, -1 is an eigenvalue, and the orthogonal line is its eigenspace.

The characteristic polynomial, the main tool for finding eigenvalues. How do you find what values the eigenvalues λ can be? In the finite dimensional case, it comes down to finding the roots of a particular polynomial, called the characteristic polynomial.

Suppose that λ is an eigenvalue of A. That means there is a nontrivial vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. Equivalently, $A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0}$, and we can rewrite that as $(A - \lambda I)\mathbf{x} = \mathbf{0}$, where I is the identity matrix. But a homogeneous equation like $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\mathbf{x}$ if and only if the determinant of $A - \lambda I$ is 0. We've shown the following theorem.

Theorem 5. A scalar λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$. In other words, λ is a root of the polynomial $\det(A - \lambda I)$, which we call the *characteristic polynomial* or *eigenpolynomial*. The equation $\det(A - \lambda I) = 0$ is called the *characteristic equation* of A.

Note that the characteristic polynomial has degree n. That means that there are at most n eigenvalues. Since some eigenvalues may be repeated roots of the characteristic polynomial, there may be fewer than n eigenvalues.

We can use characteristic polynomials to give an alternate proof that conjugate matrices have the same eigenvalues. Suppose that $B = P^{-1}AP$. We'll show that the characteristic polynomials of A and B are the same, that is,

$$\det(A - \lambda I) = \det(B - \lambda I).$$

That will imply that they have the same eigenvalues.

$$\begin{aligned} \det(B - \lambda I) &= \det(P^{-1}AP - \lambda I) \\ &= \det(P^{-1}AP - P^{-1}\lambda I P) \\ &= \det(P^{-1}(A - \lambda I)P) \\ &= \det(P^{-1})\,\det(A - \lambda I)\,\det(P) \\ &= \det(A - \lambda I) \end{aligned}$$

How to find eigenvalues and eigenspaces. Now we know the eigenvalues are the roots of the

characteristic polynomial. We'll illustrate this with an example. Here's the process to find all the eigenvalues and their associated eigenspaces.

1). Form the characteristic polynomial

$$\det(A - \lambda I).$$

2). Find all the roots of it. Since it is an nth degree polynomial, that can be hard to do by hand if n is very large. Its roots are the eigenvalues $\lambda_1, \lambda_2, \ldots$.

3). For each eigenvalue λ_i, solve the mattrix equation $(A - \lambda_i I)\mathbf{x} = \mathbf{0}$ to find the λ_i-eigenspace.

Example 6. We'll find the characteristic polynomial, the eigenvalues and their associated eigenvectors for this matrix:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 3 & 0 \\ 3 & 2 & 2 \end{bmatrix}$$

The characteristic polynomial is

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 1-\lambda & 0 & 0 \\ -3 & 3-\lambda & 0 \\ 3 & 2 & 2-\lambda \end{vmatrix} \\ &= (1-\lambda)(3-\lambda)(2-\lambda). \end{aligned}$$

Fortunately, this polynomial is already in factored form, so we can read off the three eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 3$, and $\lambda_3 = 2$. (It doesn't matter the order you name them.) Thus, the spectrum of this matrix is the set $\{1, 2, 3\}$.

Let's find the λ_1-eigenspace. We need to solve $A\mathbf{x} = \lambda_1\mathbf{x}$. That's the same as solving $(A - \lambda_1 I)\mathbf{x} = \mathbf{0}$. The matrix $A - \lambda_1 I$ is

$$\begin{bmatrix} 0 & 0 & 0 \\ -3 & 2 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

which row reduces to

$$\begin{bmatrix} 1 & 0 & \frac{1}{6} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}$$

and from that we can read off the general solution

$$(x, y, z) = (-\tfrac{1}{6}z, -\tfrac{1}{4}z, z)$$

where z is arbitrary. That's the one-dimensional 1-eigenspace (which consists of the fixed points of the transformation).

Next, find the λ_2-eigenspace. The matrix $A - \lambda_2 I$ is

$$\begin{bmatrix} -2 & 0 & 0 \\ -3 & 0 & 0 \\ 3 & 2 & -1 \end{bmatrix}$$

which row reduces to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

and from that we can read off the general solution

$$(x, y, z) = (0, \tfrac{1}{2}z, z)$$

where z is arbitrary. That's the one-dimensional 3-eigenspace.

Finally, find the λ_3-eigenspace. The matrix $A - \lambda_3 I$ is

$$\begin{bmatrix} -1 & 0 & 0 \\ -3 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix}$$

which row reduces to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and from that we can read off the general solution

$$(x, y, z) = (0, 0, z)$$

where z is arbitrary. That's the one-dimensional 2-eigenspace.

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