Where dimension doesn't work
Math 130 Linear Algebra

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Dimension works fine for vector spaces over fields, but there are things analogous to vector spaces over fields where it doesn't work. If you look at those things, you'll get a better idea of how nice things are for vector spaces.

Rings. A ring is like a field except division is not required for a ring. A *commutative ring with identity*, or what we'll call simply a *ring*, is a set equipped with two binary operations that satisfy all the axioms for fields except one. There no requirement that multiplication have inverses of nonzero elements. There may elements x in a ring for which there is no such thing as $\frac{1}{x}$.

Fields, of course, are special cases of rings, but there are a lot of important rings that are studied extensively.

The most important ring is the the ring of integers $\mathbf{Z}$ which contains all positive and negative whole numbers as well as 0. In $\mathbf{Z}$, the only elements that have inverses are 1 and -1. All the rest, like 2, have no inverses in $\mathbf{Z}$. By the way, an element in a ring that does have an inverse is called a *unit*.

Another important ring is the ring of polynomials over a field F. The ring of polynomials in the variable x with coefficients in F is denoted $F[x]$. The units in $F[x]$ are the nonzero constant polynomials.

Rings are studied extensively in our modern algebra course.

Modules over a ring, and Abelian groups. In the same way you can have vector spaces over a field F, you can have modules over a ring R. In fact, the eight axioms for a *module* over a ring are identical to the eight axioms for a vectorspace over a field, except that the scalars are taken in the ring R instead of in a field.

For example, $\mathbf{R}[x] \times \mathbf{R}[x]$ is a module over the ring $\mathbf{R}[x]$.

The axioms for modules over $\mathbf{Z}$ can be simplified since scalar multiplication by elements of $\mathbf{Z}$ can be reduced to addition. For example, $3a = a + a + a$. Because of that, we don't need to specify all eight axioms for modules over $\mathbf{Z}$. The first four involving addition are enough. Since the axioms can be given entirely in terms of addition, modules over $\mathbf{Z}$ have another name, *Abelian groups*. In other words, an Abelian group is a set equipped with an operation called addition that satisfies these four axioms

1. Addition is commutative: $v + w = w + v$ for all elements v and w;
2. Addition is associative: $(u+v)+w = u+(v+w)$ for all elements u, v, and w;
3. There is an element, denoted 0 and called *zero*, such that $v + 0 = v = 0 + v$ for each element v; and
4. For each element v, there is another element, denoted $-v$ and called the *negation* of v, such that $v + (-v) = 0$.

Examples of Abelian groups include $\mathbf{Z}_n$, the integers modulo n. (These are all rings as well if you include multiplication, but if you only consider them with the operation of addition, then they're Abelian groups.)

Where dimension doesn't work. It generally doesn't work for modules over a ring. In fact, it doesn't work for Abelian groups.

For a small example take $\mathbf{Z}_6$, the integers modulo 6. It's got 6 elements which we can denote 0, 1, 2, 3, 4, and 5. Addition is modulo 6, so, for example, $3 + 5 = 2$.

Using our definition of spanning sets that we used for vector spaces, we can find two different size minimal spanning sets. One spanning set is

just $S = \{1\}$ since every elements in $\mathbf{Z}_6$ is a integral multiple of 1. Another spanning set is $T = \{b_1, b_2\} = \{2, 3\}$. Every element in $\mathbf{Z}_6$ is a linear combination of $b_1 = 2$ and $b_2 = 3$. For example, $4 = 2b_2$, $5 = b_2 + b_3$, and $1 = 2b_2 + b_3$. But neither b_1 nor b_2 is a multiple of the other. Both are required to span Z_6.

Thus, the concept of dimension doesn't work here.

There is a classification of Abelian groups, but it's more complicated and depends on other concepts. For modules over other rings than $\mathbf{Z}$, the situation is even more complicated.

So why does dimension work for vector spaces? Well, it's got to be because of division somehow, and if you look at the proof of the replacement theorem that we used to prove that dimension works for vector spaces, you'll see that we used the fact that if we had dependent vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

then any vector with a nonzero coefficient can be found as a linear combination of the others, and that required division by that nonzero coefficient.

In a vector space, one element being a linear combination of others is equivalent to $\mathbf{0}$ being a nontrivial linear combination of them all. That equivalence requires division.

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