Kernel, image, nullity, and rank continued Math 130 Linear Algebra

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We discussed the rank and nullity of a linear transformation earlier. Let's summarize that discussion and emphasize what it means in terms of matrices.

Definitions. For $V \xrightarrow{T} W$ a linear transformation, the *kernel* or *null space* of T is $\ker(T) = T^{-1}(\mathbf{0})$, while *image* or *range* of T is $\operatorname{im}(T) = T(V)$. The *nullity* of T is the dimension of its kernel while the *rank* of T is the dimension of its image. These are denoted $\operatorname{nullity}(T)$ and $\operatorname{rank}(T)$, respectively.

Given coordinate systems for V and W, so that every linear transformation T can be described by a matrix A so that $T(\mathbf{x}) = A\mathbf{x}$.

We define the *kernel*, *image*, *rank*, and *nullity* of an $m \times n$ matrix A as the rank of the corresponding linear transformation $F^n \to F^m$.

We showed directly from the definitions some facts concerning ranks and nullity of compositions. We showed that $\operatorname{nullity}(T) \leq \operatorname{nullity}(U \circ T)$ and $\operatorname{rank}(U \circ T) \leq \operatorname{rank}(U)$. We can state that result in terms of matrices.

Theorem 1. Given matrices A and B that can be multiplied together,

$$\operatorname{nullity}(B) \leq \operatorname{nullity}(AB)$$

and

$$\operatorname{rank}(AB) \leq \operatorname{rank}(A).$$

The dimension theorem and its corollaries. The dimension theorem said that for a linear transformation $V \xrightarrow{T} W$, if the domain V is finite dimensional, then

$$\dim(V) = \operatorname{rank}(T) + \operatorname{nullity}(T).$$

We can translate this as a theorem on matrices where the matrix A represents the transformation T.

Theorem 2 (Dimension theorem for matrices)**.** For an $m \times n$ matrix A

$$n = \operatorname{rank}(A) + \operatorname{nullity}(A).$$

We showed that a linear transformation $V \xrightarrow{T} W$ was one-to-one if and only if its $\operatorname{nullity}(T) = 0$. Thus,

Theorem 3. A matrix A represents a one-to-one transformation if and only if $\operatorname{nullity}(A) = 0$.

We also showed that a linear transformation $V \xrightarrow{T} W$ was an isomorphism if and only if $\operatorname{nullity}(T) = 0$ and $\operatorname{rank}(T) = \dim(W)$. In that case, $\dim(V) = \dim(W)$. This yields the following theorem for matrices.

Theorem 4. An $n \times n$ matrix A has an inverse if and only if either of the following equivalent conditions holds: $\operatorname{rank}(A) = n$ or $\operatorname{nullity}(A) = 0$.

Row and column spaces. Let A be an $m \times n$ matrix. It represents a linear transformation $F^n \to F^m$.

Our definition of the rank of a matrix A implicitly refers to the columns of A. The ith column of A is the image of the basis vector $\mathbf{e}_i$ in F^m. For example, if A is the 2×3 matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Then the second column gives the image of $\mathbf{e}_2 = (0, 1, 0)$:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$$

The rank of A is the dimension of the image in F^n, and that image is spanned by the vectors $A\mathbf{e}_i$, that is, by the columns of A.

For that reason, the image of A, $\mathrm{im}(A)$, is often called the *column space* of A; it's the subspace of F^m spanned by the columns of A.

What about the *row space* of A, the subspace of F^n spanned by the rows of A? We'll look at that next, and we'll find that the row space and the column space have the same dimension. (In some textbooks, the dimensions of these two vector spaces are called the column rank and the row rank, then they show the two ranks are equal.)

Composition with isomorphisms doesn't change the rank. Let's first look at this in terms of linear transformations, then translate our results into statements about matrices.

Theorem 5. Composing a linear transformation with an isomorphism doesn't change rank.

Proof. Let $V \xrightarrow{T} W$ be a linear transformation, and let $V' \xrightarrow{R} V$ and $W \xrightarrow{S} W'$ be isomorphisms. Then we'll show that the two compositions $T \circ R : V' \xrightarrow{R} V \xrightarrow{T} W$ and $S \circ T : V \xrightarrow{T} W \xrightarrow{S} W'$ have the same rank that T has.

For the first one, every vector in V is in the image of R, so the image of $T \circ R$ is identical to the image of T, so they have the same rank.

For the second one, the isomorphism $W \xrightarrow{S} W'$ restricts to an isomorphism $\mathrm{im}(T) \to \mathrm{im}(S \circ T)$ since it's a one-to-one correspondence that preserves the vector space operations. Therefore $\mathrm{im}(T)$ and $\mathrm{im}(S \circ T)$ have the same dimension. Thus, $\mathrm{rank}(T) = \mathrm{rank}(S \circ T)$. Q.E.D.

In terms of matrices, this says if you multiply a matrix by an invertible square matrix, either on the left or on the right, the resulting matrix has the same rank.

Theorem 6. The elementary row operations and the corresponding elementary column operations on a matrix preserve the rank of a matrix.

Proof. An elementary row operation multiplies a matrix by an elementary matrix on the left. Those elementary matrices are invertible, so the row operations preserve rank.

Elementary column operations will multiply a matrix by an elementary matrix on the right, so they'll preserve rank, too. Q.E.D.

By means of elementary row and column operations, you can always transform a matrix to one in which there are only 0's and 1's, and the 1's, if there are any, start at the upper left corner and are placed down and to the right, for example

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The transformed matrix has the same rank as the original one, and its rank is equal to the number of 1's in it. That can be read either as the number of nonzero rows, or as the number of nonzero columns. From this observation, we can derive the following theorem.

Theorem 7. The rank of a matrix is equal to the rank of its transpose. In other words, the dimension of the column space equals the dimension of the row space, and both equal the rank of the matrix.

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