CLARK UNIVERSITY 1887

Vectors
Math 130 Linear Algebra
D Joyce, Fall 2013

Welcome to the course! Much of today we'll discuss the administration of the course. The course web page is at

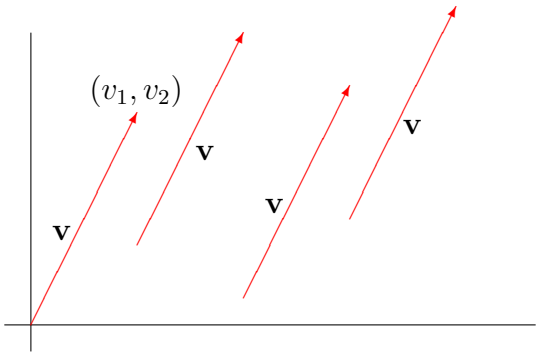
`http://math.clarku.edu/~djoyce/ma130/`

We will discuss the general description of the course, its goals and objectives, the syllabus, the textbook, assignments, quizzes, tests, course grade, and various sundry items.

Vectors. After discussing the course in general, we'll move on the subject matter. We're starting out with something you may already know, vectors. We'll mostly illustrate vectors in the plane $\mathbf{R}^2$, but everything we do will apply to vectors in the higher dimensional spaces $\mathbf{R}^n$. We'll look at what they are, their addition, subtraction, and multiplication by scalars (i.e. real numbers), and a graphical interpretation of these vector operations.

Vectors in the plane $\mathbf{R}^2$. A vector $\mathbf{v}$ can be interpreted as an arrow in the plane $\mathbf{R}^2$ with a certain length and a certain direction. The same vector can be moved around in the plane if you don't change its length or direction. Thus, one vector can be interpreted as lots of different arrows with a given length and direction.

When a vector is put in "standard position," the head of the arrow is at a point (v_1, v_2) and the tail of the arrow is at the origin $(0, 0)$. Using this standard position, we identify a vector $\mathbf{v}$ with the point (v_1, v_2). Thus, sometimes we say "the vector (v_1, v_2)" deliberately confusing the point (v_1, v_2) with the vector $\mathbf{v}$. We'll call $\mathbf{R}^2$ the vector space of two dimensions.



Note that boldface symbols are often for vectors like we did for $\mathbf{v}$, but italic for numbers like v_1 and v_2. This isn't necessary, but using different fonts and cases for the different kinds of mathematical things that we consider helps to tell us what's what.

The zero vector, $\mathbf{0} = (0, 0)$ is special since it's the only vector that has no direction. It has 0 length. We represent it as a point, in fact, any point, since it can be moved around in the plane.

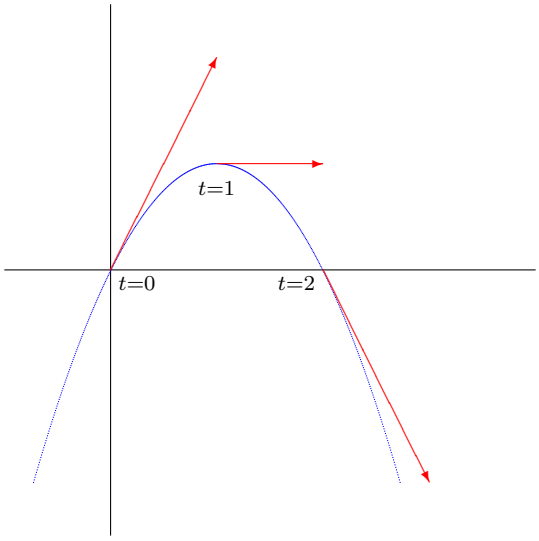
When we think of the vector $\mathbf{v}$ as an arrow somewhere else in the plane, it is an arrow parallel to the standard arrow and with the same length. Then it can be any arrow with head (c, d) and tail (a, b) so long as $c - a = v_1$ and $d - b = v_2$. We'll say the vector $\mathbf{v} = (c - a, d - b)$ is the *displacement vector* from the point (a, b) to the point (c, d).

Vectors in space $\mathbf{R}^3$. A vector in space also has a certain length and direction. You can move it around in space if you don't change its length or direction. It can be placed in standard position with its tail at the origin $(0, 0, 0)$ and its head at a point with three coordinates (v_1, v_2, v_3). We'll give the vector those coordinates so that $\mathbf{v} = (v_1, v_2, v_3)$.

Most of the time we'll look at vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$, but the basic concepts generalize to $\mathbf{R}^n$ where n is any positive number. That allows us to work with n-dimensional space easily even though we can't visualize it.

An application of vectors: velocity vectors. You may have already seen this when you studied motion of a object in more than one dimension, perhaps described as parametric equations.

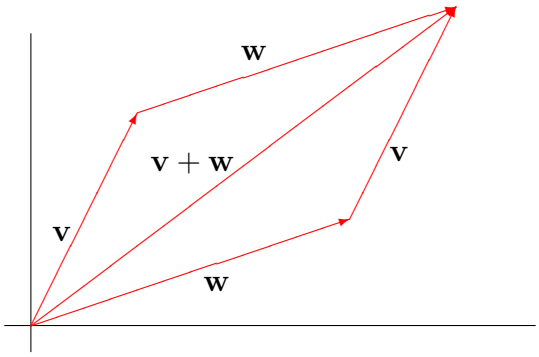
Suppose that at time t a moving point is at the location (x, y) where both x and y re functions of t. Take, for example $(x, y) = (t, 2t - t^2)$. The point travels some curve, called its trajectory, in this example, the parabola $y = 2x - x^2$. The velocity of the point is given by the derivative $(x', y') = (1, 2 - 2t)$. You can draw the velocity vector (x', y') at a given time with its tail at the position (x, y) at that time. It will be tangent to the trajectory. At $t = 0$, $(x, y) = (0, 0)$ and $(x', y') = (1, 2)$. At $t = 1$, $(x, y) = (1, 2)$ and $(x', y') = (1, 0)$. At $t = 2$, $(x, y) = (1, 0)$ and $(x', y') = (1, -2)$.



The direction of the velocity vector (x', y') tells you which way the point is moving while the length of that vector tells you how fast it's going. The length of the velocity vector is called *speed*.

In this course we'll be concentrating on the concept of vectors themselves, but we'll look at a some applications of linear algebra as well.

Geometric interpretation of addition of vectors. When you interpret vectors as arrows, there are a couple of ways to interpret addition. One is by what is called the "parallelogram law." In order to see the sum $\mathbf{v} + \mathbf{w}$ of two vectors $\mathbf{v}$ and $\mathbf{w}$, draw both $\mathbf{v}$ and $\mathbf{w}$ in their standard positions with their tails at the origin. Then translate $\mathbf{v}$ so that its tail is at the head of $\mathbf{w}$ to get another arrow which can also be labelled $\mathbf{v}$. Likewise, translate $\mathbf{w}$ so that its tail is at the head of the original arrow $\mathbf{v}$ to get another arrow labelled $\mathbf{w}$. That completes a parallelogram with two sides being parallel $\mathbf{v}$ arrows and the other two sides parallel $\mathbf{w}$ arrows.



The arrow which goes from the origin to the opposite corner of the parallelogram is the vector $\mathbf{v} + \mathbf{w}$. Thus, you can see $\mathbf{v} + \mathbf{w}$ either as the arrow $\mathbf{v}$ followed by the arrow $\mathbf{w}$, or as the arrow $\mathbf{w}$ followed by the arrow $\mathbf{v}$.

We can use this parallelogram to determine coordinates for the sum $\mathbf{v} + \mathbf{w}$.

$$\mathbf{v} + \mathbf{w} = (v_1, v_2) + (w_1, w_2) = (v_1 + w_1, v_2 + w_2).$$

In words, the coordinates of the sum of two vectors are the sums of the coordinates of the vectors. That is, sums of vectors are computed coordinatewise.

Once we've got a geometric interpretation for addition, we automatically get geometric interpretations for negation and subtraction. The negation $-\mathbf{v}$ of a vector $\mathbf{v}$ is an arrow that points in the opposite direction of $\mathbf{v}$. That implies that

$$-\mathbf{v} = -(v_1, v_2) = (-v_1, -v_2).$$

Thus, negations are computed coordinatewise. Likewise, negation is computed coordinatewise.

$$\mathbf{v} - \mathbf{w} = (v_1, v_2) - (w_1, w_2) = (v_1 - w_1, v_2 - w_2)$$

Multiplication of vectors by scalars. Another operation on vectors we can perform is to multiply a vector by a real number. We'll use the term *scalar* as a synomym for real number when our vectors are real vectors. (Sometimes other fields of scalars are used. If the coordinates of the vectors are complex numbers, then the scalars are complex rather than real).

For example we can double a vector $\mathbf{v}$ to get the vector $2\mathbf{v}$, which is $\mathbf{v}+\mathbf{v}$. Coordinatewise, it's $2\mathbf{v} = 2(v_1, v_2) = (2v_1, 2v_2)$. Likewise we can multiply a vector $\mathbf{v}$ by any scalar c to get $c\mathbf{v} = c(v_1, v_2) = (cv_1, cv_2)$.

If the scalar c is greater than 1, then stretch $\mathbf{v}$ by a factor of c to get $c\mathbf{v}$. If c is between 0 and 1, then squeeze $\mathbf{v}$ by that factor. If c is negative, then $c\mathbf{v}$ points in the opposite direction

of $\mathbf{v}$ stretched or squeezed by the appropriate factor. Of course, $1\mathbf{v} = \mathbf{v}$, $-1\mathbf{v} = -\mathbf{v}$, and $0\mathbf{v} = \mathbf{0}$.

You can also divide a vector by a scalar, but we won't treat it as a separate operation. Instead, we'll take it to be multiplication by the reciprocal of the scalar. So, for example, instead of $\mathbf{v}/3$, we'll use $\frac{1}{3}\mathbf{v}$.

We say one vector is in the *same direction* as another if each is a positive multiple of the other. They're in the *opposite direction* if each is a negative multiple of the other.

Properties of vector operations. The expected properties of the operations on vectors hold. For example, addition is commutative, $\mathbf{v}+\mathbf{w} = \mathbf{w}+\mathbf{v}$, and it's associative, $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{w}+\mathbf{v})$, so when you have several vectors to add, you can add them in any order.

The zero vector acts like it should, $\mathbf{v} + \mathbf{0} = \mathbf{v}$. Negation works right, $-\mathbf{v} + \mathbf{v} = \mathbf{0}$. Subtraction works right, too. It's the inverse of addition: $\mathbf{u}+\mathbf{v} = \mathbf{w}$ if and only if $\mathbf{u} = \mathbf{w}-\mathbf{v}$.

Furthermore, multiplication by scalars has the expected properties. It's associative, $(bc)\mathbf{v} = b(c\mathbf{v})$, so we won't have to write extra parentheses. It distributes over addition on the left, $(b+c)\mathbf{v} = b\mathbf{v} + c\mathbf{v}$, and on the right $c(\mathbf{v}+\mathbf{w}) = c\mathbf{v} + c\mathbf{w}$.

Because of all these properties, you can use much of what you know about ordinary algebra for vector algebra, too.

Note that we don't have multiplication of two vectors. Later in the course, we'll look at two different kinds of multiplcation, one being 'inner products' which are also called 'dot products', and the other being 'outer products', also called 'cross products'.

n-dimensional space $\mathbf{R}^n$. We can't visualize space beyond three dimensions, but by means of algebra we can work with vectors in $\mathbf{R}^n$ just as easily as vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$. There are many applications of n-dimensional vectors, but they're not the same physical applications of 2- and 3-dimensional vectors. We'll use a similar notation for them. An n-vector $\mathbf{v}$ will have n coordinates and be written $\mathbf{v} = (v_1, v_2, \ldots, v_n)$. The operations of addition, subtraction, and scalar multiplication work coordinatewise just as they do in $\mathbf{R}^2$ and $\mathbf{R}^3$, so

$$\begin{aligned} \mathbf{v}+\mathbf{w} &= (v_1, v_2, \ldots, v_n) + (w_1, w_2, \ldots, w_n) = (v_1+w_1, v_2+w_2, \ldots, v_n+w_n), \text{ and} \\ c\mathbf{v} &= c(v_1, v_2, \ldots, v_n) = (cv_1, cv_2, \ldots, cv_n). \end{aligned}$$

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