First Test Answers
Math 120 Calculus I
September, 2013

Scale. 90–100 A, 80–89 B, 65–79 C. Median 80.

1. [12] On limits of average rates of change. Let $f(x) = x^2 - 3x$.

a. [4] Write down an expression that gives the average rate of change of this function over the interval between x and $x+h$, and simplifly the expression.

$$\frac{f(x+h)-f(x)}{h}$$
$$= \frac{((x+h)^2 - 3(x+h)) - (x^2-3x)}{h}$$

b. [8] Compute the limit as $h \to 0$ of that average rate of change.

$$\lim_{h\to 0} \frac{((x+h)^2 - 3(x+h)) - (x^2-3x)}{h}$$
$$= \lim_{h\to 0} \frac{x^2+2xh+h^2-3x-3h-x^2+3x)}{h}$$
$$= \lim_{h\to 0} \frac{2xh+h^2-3h)}{h}$$
$$= \lim_{h\to 0}(2x+h-3) = 2x-3$$

2. [10; 5 points each] On the intutitive concept of limit and continuity.

a. [5] Sketch the graph $y = f(x)$ of a function for which $\lim_{x\to 0} f(x)$ does not exist.

There are many such graphs. For example, if there's a jump in the value of f at $x=0$, then that limit won't exist. See section 2.4 of the text.

b. [5] Sketch the graph $y = f(x)$ of a function defined everywhere, the limit $\lim_{x\to 0} f(x)$ does exist, but f is not continuous at $x=0$.

This can be achieved by making $f(0)$ unequal to the limit, but make sure that the function is defined at $x=0$. See section 2.5 of the text.

3. [10; 5 points each property] On asymptotes.

a. Sketch the graph of a function f such that

$$\lim_{x\to 2^-} f(x) = \infty \text{ and } \lim_{x\to 2^+} f(x) = -\infty.$$

The graph of the function should should be asymptotic to the vertical line $x=2$. See section 2.6 of the text.

b. Sketch the graph of a function f such that $\lim_{x\to\infty} f(x) = 1$.

The graph of the function should should be asymptotic to the horizontal line $y=1$. See section 2.6 of the text.

4. [28; 7 points each part] Evaluate the following limits. If a limit diverges to $\pm\infty$ it is enough to say that it doesn't exist.

a. $\displaystyle\lim_{x\to 1} \frac{x^2-1}{x^2-3x+2}$

The expression needs to be simplified before taking the limit.

$$\lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)(x-2)} = \lim_{x\to 1} \frac{x+1}{x-2} = -2$$

b. $\displaystyle\lim_{x\to 1} \frac{x^2-4}{x^2-3x+2}$

The number approaches -3 while the denominator approaches 0, so the limit of the quotient doesn't exist.

c. $\displaystyle\lim_{x\to\infty} \frac{4x^3-2x}{9x^3+1}$

The numerator and denominator have the same degree, so as $x\to\infty$, the value approaches the ratio of the leading coefficients, $\frac{4}{9}$. This can be seen by dividing the numerator and denominator by x^3

$$\lim_{x\to\infty} \frac{4x^3-2x}{9x^3+1} = \lim_{x\to\infty} \frac{4-2/x^2}{9+1/x^3}$$
$$= \frac{4-0}{9-0} = \frac{4}{9}$$

d. $\lim_{x\to 0} \dfrac{4\sin x}{5x}$.

Recall that $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$. Therefore this limit equals $\frac{4}{5}$.

5. [15] On the formal definition of limit.

Consider the limit $\lim_{x\to 5}(2x-3)$ which, of course, has the value 7. Since it has the value 7, that means that for each $\epsilon > 0$, there exists some $\delta > 0$, such that for all x, if $0 < |x-5| < \delta$, then $|(2x-3)-7| < \epsilon$.

Let $\epsilon = \frac{1}{2}$. Find a value of δ that works for this ϵ.

You need to find a value of δ so that

$$0 < |x-5| < \delta \text{ implies } |(2x-3)-7| < \tfrac{1}{2}.$$

The expression $|(2x-3)-7|$ can be rewritten as $|2x-10|$ which equals $2|x-5|$. Therefore, the condition $|(2x-3)-7| < \frac{1}{2}$ is equivalent to $|x-5| < \frac{1}{4}$. Thus, you need to find a value of δ so that

$$0 < |x-5| < \delta \text{ implies } |x-5| < \tfrac{1}{4}.$$

Such a value is $\delta = \frac{1}{4}$.

6. [10] Suppose that θ is an angle between $-\pi/2$ and 0, and that $\cos\theta = \frac{1}{2}\sqrt{2}$. Determine the value of $\sin\theta$.

Since $\cos\theta = \frac{1}{2}\sqrt{2}$, the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ implies $\sin^2\theta + \frac{1}{2} = 1$. Hence, $\sin^2\theta = \frac{1}{2}$, so $\sin\theta = \pm\frac{1}{2}\sqrt{2}$. Since θ is an angle between $-\pi/2$ and 0, the sine of θ is negative. Thus $\sin\theta = -\frac{1}{2}\sqrt{2}$.

7. [15; 5 points each part] Suppose that $\lim_{x\to\pi} f(x) = 5$ and $\lim_{x\to\pi} g(x) = 3$. Evaluate each of the following limits, or explain why it doesn't exist

a. $\lim_{x\to\pi} \dfrac{f(x)}{g(x)}$

Since $f(x)$ approaches 5, and $g(x)$ approaches 3, the quotient approaches $\frac{5}{3}$.

b. $\lim_{x\to\pi} \dfrac{f(x)}{g(x) + 3\cos x}$

As x approaches π, $\cos x$ approaches -1. Therefore the denominator approaches 0. But the numerator approaches 5, so the limit doesn't exist.

c. $\lim_{x\to\pi} \sqrt{x + f(x)g(x)}$

The product $f(x)g(x)$ approaches 15, so $x + f(x)g(x)$ approaches $\pi + 15$. Since the square root function is continuous, the limit approaches $\sqrt{\pi + 15}$.