CLARK UNIVERSITY 1887

First Test Answers
Math 120 Calculus I
September, 2013

Scale. 90100 A, 8089 B, 6579 C. Median 80.

1. [12] On limits of average rates of change. Let $f(x) = 5x^2 + 4$.

a. [4] Write down an expression that gives the average rate of change of this function over the interval between x and $x + h$, and simplifly the expression.

$$\begin{aligned} & \frac{f(x+h) - f(x)}{h} \\ = & \frac{(5(x+h)^2 + 4) - (5x^2 + 4)}{h} \end{aligned}$$

b. [8] Compute the limit as $h \to 0$ of that average rate of change.

$$\begin{aligned} & \lim_{h\to 0} \frac{(5(x+h)^2 + 4) - (5x^2 + 4)}{h} \\ = & \lim_{h\to 0} \frac{5x^2 + 10xh + 5h^2 + 4 - 5x^2 - 4)}{h} \\ = & \lim_{h\to 0} \frac{10xh + 5h^2}{h} \\ = & \lim_{h\to 0}(10x + 5h) = 10x \end{aligned}$$

2. [10; 5 points each] On the intutitive concept of limit and continuity.

a. [5] Sketch the graph $y = f(x)$ of a function for which $\lim_{x\to 3} f(x)$ does not exist.

There are many such graphs. For example, if there's a jump in the value of f at $x = 3$, then that limit won't exist. See section 2.4 of the text.

b. [5] Sketch the graph $y = f(x)$ of a function defined everywhere, the limit $\lim_{x\to 3} f(x)$ does exist, but f is not continuous at $x = 3$.

This can be achieved by making $f(3)$ unequal to the limit, but make sure that the function is defined at $x = 3$. See section 2.5 of the text.

3. [10; 5 points each property] On asymptotes.

a. Sketch the graph of a function f such that

$$\lim_{x\to 2^-} f(x) = 0 \text{ and } \lim_{x\to 2^+} f(x) = -\infty.$$

The graph of the function should should be asymptotic to the vertical line $x = 2$. See section 2.6 of the text.

b. Sketch the graph of a function f such that $\lim_{x\to -\infty} f(x) = 1$.

The graph of the function should should be asymptotic to the horizontal line $y = 1$. See section 2.6 of the text.

4. [28; 7 points each part] Evaluate the following limits. If a limit diverges to $\pm\infty$ it is enough to say that it doesn't exist.

a. $\lim_{x\to 1} \frac{x^2 - 4x + 4}{x^2 - x - 2}$

As x approaches 1, the numerator approaches 1 while the denominator approaches -2, so the quotient approaches $-\frac{1}{2}$.

b. $\lim_{x\to 2} \frac{x^2 - 4x + 4}{x^2 - x - 2}$

The expression needs to be simplified before taking the limit.

$$\lim_{x\to 2} \frac{(x+2)(x-2)}{(x+1)(x-2)} = \lim_{x\to 2} \frac{x+2}{x+1} = \frac{4}{3}$$

c. $\lim_{x\to\infty} \frac{3x^2 - 2x + 1}{9x^3 + x}$

The numerator has a lower degree than the denominator, so as $x \to \infty$, the limit approaches 0. This can be seen by dividing the numerator and denominator by x^3

$$\begin{aligned} \lim_{x\to\infty} \frac{3x^2 - 2x + 1}{9x^3 + x} &= \lim_{x\to\infty} \frac{3/x - 2/x^2 + 1/x^3}{9 + 1/x^2} \\ &= \frac{0 - 0 + 0}{9 + 0} = 0 \end{aligned}$$

d. $\lim_{x\to 0} \frac{5x}{8\sin x}$.

Recall that $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Therefore $\lim_{x\to 0} \frac{x}{\sin x} = 1$, hence this limit equals $\frac{5}{8}$.

5. [15] On the formal definition of limit.

Consider the limit $\lim_{x\to 3}(9-2x)$ which, of course, has the value 3. Since it has the value 3, that means that for each $\epsilon > 0$, there exists some $\delta > 0$, such that for all x, if $0 < |x-3| < \delta$, then $|(9-2x)-3| < \epsilon$.

Let $\epsilon = \frac{1}{3}$. Find a value of δ that works for this ϵ.

You need to find a value of δ so that

$$0 < |x-3| < \delta \text{ implies } |(9-2x)-3| < \tfrac{1}{3}.$$

The expression $|(9-2x)-3|$ can be rewritten as $|6-2x|$ which equals $2|x-3|$. Therefore, the condition $|(9-2x)-3| < \frac{1}{3}$ is equivalent to $|x-3| < \frac{1}{6}$. Thus, you need to find a value of δ so that

$$0 < |x-3| < \delta \text{ implies } |x-3| < \tfrac{1}{6}.$$

Such a value is $\delta = \frac{1}{6}$.

6. [10] Suppose that θ is an angle between $\pi/2$ and π, and that $\sin\theta = \frac{1}{2}\sqrt{2}$. Determine the value of $\cos\theta$.

Since $\sin\theta = \frac{1}{2}\sqrt{2}$, the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ implies $\frac{1}{2} + \cos^2\theta = 1$. Hence, $\cos^2\theta = \frac{1}{2}$, so $\cos\theta = \pm\frac{1}{2}\sqrt{2}$. Since θ is an angle between $\pi/2$ and π, the cosine of θ is negative. Thus $\cos\theta = -\frac{1}{2}\sqrt{2}$.

7. [15; 5 points each part] Suppose that $\lim_{x\to\pi/2} f(x) = 4$ and $\lim_{x\to\pi/2} g(x) = 5$. Evaluate each of the following limits, or explain why it doesn't exist

a. $\lim_{x\to\pi/2} \frac{f(x)+g(x)}{f(x)-g(x)}$

Since $f(x)$ approaches 4, and $g(x)$ approaches 9, the numerator approaches 9 while the denominator approaches -1. Therefore quotient approaches -9.

b. $\lim_{x\to\pi/2} \frac{x}{g(x)-f(x)-\sin x}$

As x approaches $\pi/2$, $\sin x$ approaches 1. Therefore the denominator approaches 0. But the numerator approaches $\pi/2$, so the limit doesn't exist.

c. $\lim_{x\to\pi/2} \sqrt{(g(x))^2 + (f(x))^2}$

The sum $(g(x))^2 + (f(x))^2$ approaches 25. Since the square root function is continuous, the limit approaches the square root of 25, which is 5.