Proof of the Fundamental Theorem of Calculus

Math 121 Calculus II

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The statements of FTC and FTC^{-1}. Before we get to the proofs, let's first state the Fundamental Theorem of Calculus and the Inverse Fundamental Theorem of Calculus. When we do prove them, we'll prove FTC^{-1} before we prove FTC. The FTC is what Oresme propounded back in 1350.

(Sometimes FTC^{-1} is called the first fundamental theorem and FTC the second fundamental theorem, but that gets the history backwards.)

Theorem 1 (FTC)**.** If F' is continuous on $[a, b]$, then

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

In other words, if F is an antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

A common notation for $F(b) - F(a)$ is $F(x)\Big|_a^b$.

There are stronger statements of these theorems that don't have the continuity assumptions stated here, but these are the ones we'll prove.

Theorem 2 (FTC^{-1})**.** If f is a continuous function on the closed interval $[a, b]$, and F is its *accumulation* function defined by

$$F(x) = \int_a^x f(t)\,dt$$

for x in $[a, b]$, then F is differentiable on $[a, b]$ and its derivative is f, that is, $F'(x) = f(x)$ for $x \in [a, b]$.

Frequently, the conclusion of this theorem is written

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Note that a different variable t is used in the integrand since x already has a meaning. Logicians and computer scientists are comfortable using the same variable for two different purposes, but they have to resort to the concept of "scope" of a variable in order to pull that off. It's usually easier to make sure that each variable only has one meaning. Thus, we use one variable x as a limit of integration, but a different variable t inside the integral.

Our first proof is of the FTC^{-1}.

Proof of the FTC^{-1}. First of all, since f is continuous, it's integrable, that is to say,

$$F(x) = \int_a^x f(t)\,dt$$

does exist.

We need to show that $F'(x) = f(x)$. By the definition of derivatives,

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right) \\ &= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt \end{aligned}$$

We'll show that this limit equals $f(x)$. Although a complete proof would consider both cases $h < 0$ and $h > 0$, we'll only look at the case when $h > 0$; the case for $h < 0$ is similar but more complicated by negative signs.

We'll concentrate on the values of the continuous function $f(x)$ on the closed interval $[x, x+h]$. On this interval, f takes on a minimum value m_h and a maximum value M_h (by the Extremal Value Theorem for continuous functions on closed intervals). Since $m_h \le f(t) \le M_h$ for t in this interval $[x, x+h]$, therefore when we take the definite integrals on this interval, we have

$$\int_x^{x+h} m_h\,dt \le \int_x^{x+h} f(t)\,dt \le \int_x^{x+h} M_h\,dt.$$

But $\displaystyle\int_x^{x+h} m_h\,dt = hm_h$, and $\displaystyle\int_x^{x+h} M_h\,dt = hM_h$, so, dividing by h, we see that

$$m_h \le \frac{1}{h}\int_x^{x+h} f(t)\,dt \le M_h.$$

Now, f is continuous, so as $h \to 0$ all the values of f on the shortening interval $[x, x+h]$ approach $f(x)$, so, in particular, both the minimum value m_h and the maximum value M_h approach $f(x)$. But if both m_h and M_h approach the same number $f(x)$, then anything between them also approaches it, too. Thus

$$\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = f(x)$$

thereby proving $F'(x) = f(x)$. Q.E.D.

We'll now go on to prove the FTC from the FTC^{-1}.

Proof of the FTC. Let

$$G(x) = \int_a^x F'(t)\,dt.$$

Then by FTC^{-1}, $G'(x) = F'(x)$. Therefore, G and F differ by a constant C, that is, $G(x) - F(x) = C$ for all $x \in [a, b]$. But

$$G(a) = \int_a^a F'(t)\,dt = 0,$$

and $G(a) - F(a) = C$, so $C = -F(a)$. Hence, $G(x) - F(x) = -F(a)$ for all $x \in [a, b]$. In particular, $G(b) - F(b) = -F(a)$, so $G(b) = F(b) - F(a)$, that is,

$$\int_a^b F'(t)\,dt = F(b) - F(a).$$

Q.E.D.

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