Exercise from section 1.3, page 19, exercises 1, 2bdfh, 6, 8ace, 15, 23. See the answers in the book for exercises 2 and 8.

1. True/false.

a. If $V$ is a vector space and $W$ is a subset of $V$ that is a vector space, then $W$ is a subspace of $V$.

This is true if the operations on $W$ are the same ones as the operations on $V$, but false if $W$ has different operations. So it has to be considered false.

b. The empty set is a subspace of every vector space.

False. The empty set doesn’t include $0$, so it’s not.

c. If $V$ is a vector space other than the zero vector space, then $V$ contains a subspace $W$ such that $W \neq V$.

Strange statement. It’s true. Just take $W$ to be the zero subspace $0$.

d. The intersection of any two subsets of $V$ is a subspace of $V$.

Nah. The intersection of any two subspaces of $V$ is a subspace, but not any two subsets.

e. An $n \times n$ diagonal matrix can never have more than $n$ nonzero entries.

True. The only nonzero entries can be on the diagonal, and there are only $n$ entries on the diagonal.

f. The trace of a square matrix is the product of its diagonal entries.

Sum, not product.

g. Let $W$ be the $xy$-plane in $\mathbb{R}^3$; that is, $W = \{(a_1, a_2, 0) | a_1, a_2 \in \mathbb{R} \}$. Then $W = \mathbb{R}^2$.

False. They’re not equal, but isomorphic, and it does little harm to identify them.

6. Prove that $\text{tr}(aA + bB) = a \text{tr}(A) + b \text{tr}(B)$ for any $A, B \in M_{n \times n}(F)$.

You can do this either in words or symbols, your preference.

Verbal proof: The matrix $aA + bB$ has as its $i^{\text{th}}$ diagonal element $a$ times the $i^{\text{th}}$ diagonal element of $A$ plus $b$ times the $i^{\text{th}}$ diagonal element of $B$. The trace of $aA + bB$ is the sum of its diagonal elements. In that sum $a$ times each diagonal element of $A$ is added to $b$ times each diagonal element of $B$. Factoring out the $a$ and $b$, we get $a$ times the sum of the diagonal elements of $A$ plus $b$ times the sum of the diagonal elements of $B$, and that’s equal to $a \text{tr}(A) + b \text{tr}(B)$. Q.E.D.

Symbolic proof: This can be given by an extended equation where each step is either by definition of trace or by plain algebra.

$$
\text{tr}(aA + bB) = \sum_i (aA + bB)_{ii} \\
= \sum_i (aA_{ii} + bB_{ii}) \\
= a \sum_i A_{ii} + b \sum_i B_{ii} \\
= a \text{tr}(A) + b \text{tr}(B)
$$

Q.E.D.

8. Are these subspaces of $\mathbb{R}^3$?

a. $W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 | a_1 = 3a_2 \text{ and } a_3 = -a_2 \}$.

Yes, it is. The $0$ vector is in it, and it’s closed under addition and scalar multiplication. Here’s how you can show it’s closed under addition. Let $(a_1, a_2, a_3)$ and $(b_1, b_2, b_3)$ be two vectors in $W_1$. To show that $(a_1 + b_1, a_2 + b_2, a_3 + b_3)$ is also in $W_1$, That means we need so show that $a_1 + b_1 = 3(a_2 + b_2)$ and $a_3 + b_3 = -(a_2 + b_2)$. But those two equations follow by adding the given equations $a_1 = 3a_2$ and $b_1 = 3b_2$, and the equations $a_3 = -a_2$ and $b_3 = -b_2$. Showing it’s closed under scalar multiplication is easier since you only need to consider one vector and a scalar.
c. \( W_3 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 \mid 2a_1 - 7a_2 + a_3 = 0\} \).

This is a subspace, too, for the same reasons. Only the algebra is a little different. These are both examples of solution sets to a system of homogeneous linear equations, and all such solution sets form vector spaces.

e. \( W_5 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 \mid a_1 + 2a_2 - 3a_3 = 1\} \).

This is not a subspace. It’s easiest just to note that 0 is not in it, you could also show that it’s neither closed under addition nor scalar multiplication, but that’s not necessary. If it doesn’t have 0, that’s enough to know it’s not a subspace.

Although this is the solution set to a system of linear equations, because it’s not a homogeneous system, it doesn’t form a vector space.

15. Is the set \( \mathcal{D}(\mathbb{R}) \) of all differentiable functions \( \mathbb{R} \to \mathbb{R} \) a subspace of \( \mathcal{C}(\mathbb{R}) \) of all continuous functions \( \mathbb{R} \to \mathbb{R} \)?

First of all, \( \mathcal{D}(\mathbb{R}) \) is a subset of \( \mathcal{C}(\mathbb{R}) \) since every differentiable function is continuous. Next, you can check the three conditions of a subset to be a subspace of a vector space.

(a). Is the zero function 0 in \( \mathcal{D}(\mathbb{R}) \)? Yes, it’s differentiable.

(b). Is \( \mathcal{D}(\mathbb{R}) \) closed under addition? Yes, the sum of two differentiable functions is differentiable, and we know what it is by the sum rule for derivatives.

(c). Is \( \mathcal{D}(\mathbb{R}) \) closed under scalar multiplication? Yes, if \( f \) is differentiable and \( c \) is a scalar, then \((cf)' = cf'\), so \( cf \) is also differentiable.

Therefore \( \mathcal{D}(\mathbb{R}) \) is a subspace of \( \mathcal{C}(\mathbb{R}) \). Q.E.D.

On the answer sheets, I usually give just one way to do the exercises, but there are always other ways. For instance, to check if a subset is a subspace, I checked to see if 0 is in it, if it’s closed under addition, and if it’s closed under scalar multiplication. There are other ways. For instance, you could note that it’s nonempty and show it’s closed under linear combinations.

23. Let \( W_1 \) and \( W_2 \) be subspaces of a vector space \( V \).

a. Prove that the sum

\[
W_1 + W_2 = \{x + y \mid x \in W_1 \text{ and } y \in W_2\}
\]

is a subspace of \( V \) that contains both \( W_1 \) and \( W_2 \).

It’s easy to see that \( W_1 + W_2 \) contains \( W_1 \) since every element \( x \in W_1 \) equals \( x + 0 \) which lies in \( W_1 \) + \( W_2 \). Likewise \( W_1 + W_2 \) contains \( W_2 \).

To show that \( W_1 + W_2 \) is a subspace of \( V \), check the three conditions.

(a). Is \( 0 \in W_1 + W_2 \)? Yes, it equals \( 0 + 0 \).

(b). Is \( W_1 + W_2 \) closed under addition? Suppose that both \( x + y \) and \( v + w \) are in \( W_1 + W_2 \) where \( x \in W_1 \), \( y \in W_2 \), \( v \in W_1 \), and \( w \in W_2 \). Then their sum equals \((x + v) + (y + w)\) where \((x + v) \in W_1 \) and \((y + w) \in W_2 \), so the sum is in \( W_1 + W_2 \). Therefore it’s closed under addition.

(c). Is \( W_1 + W_2 \) closed under scalar multiplication? Suppose that \( x + y \) is in \( W_1 + W_2 \) with \( x \in W_1 \) and \( y \in W_2 \), and that \( c \in F \). Then \( cx \in W_1 \) and \( cy \in W_2 \), so \( c(x + y) \), which equals \( cx + cy \), lies in \( W_1 + W_2 \). Thus, it’s closed under scalar multiplication.

Therefore \( W_1 + W_2 \) is a subspace of \( V \). Q.E.D.

b. Prove that any subspace of \( V \) that contains both \( W_1 \) and \( W_2 \) must also contain \( W_1 + W_2 \).

Let \( W \) be a subspace of \( V \) that contains both \( W_1 \) and \( W_2 \). We’ll show \( W \) also contains \( W_1 + W_2 \).

Let \( x + y \) be in \( W_1 + W_2 \) with \( x \in W_1 \) and \( y \in W_2 \). Then \( x \in W \) since \( W_1 \subseteq W \), and \( y \in W \) since \( W_2 \subseteq W \). Therefore, their sum \( x + y \) also lies in \( W \). Since every element of \( W_1 + W_2 \) is in \( W \), therefore \( W \) contains \( W_1 + W_2 \). Q.E.D.

Together, parts a and b of this exercise prove this theorem: the sum of two subspaces is the smallest subspace that contains them both.

Math 130 Home Page at http://math.clarku.edu/~djoyce/ma130/